A UNITED STATES DEPARTMENT OF COMMERCE PUBLICATION

SECTION B

JOURNAL OF RESEARCH

of the NATIONAL BUREAU OF STANDARDS

JNBBAU 77B (1&2) 1-71, Jan.-June 1973

# Mathematical Sciences

U.S. DEPARTMENT OF COMMERCE
National Bureau of Standards

# Journal of Research

**of the**

**National Bureau of Standards**

**B. Mathematical Sciences**

Jan.–June • 1973

VOLUME • 77B

NUMBERS • 1 & 2

Editor: Morris Newman

Associate Editor: F. W. Olver

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1973

J. Res. Nat. Bur. Stand. (U.S.), B


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The Secretary of Commerce has determined that the publication of this periodical is necessary in the transaction of the public business required by law of this Department. Use of funds for printing this periodical has been approved by the Director of the Office of Management and Budget through June 30, 1976.

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 1 & 2, January–June 1973

# Some Hamiltonian Results in Powers of Graphs*

**Arthur M. Hobbs****

(November 16, 1972)

In this paper we show that the connectivity of the $k$th power of a graph of connectivity $m$ is at least $km$ if the $k$th power of the graph is not a complete graph. Also, we prove that removing as many as $k-2$ vertices from the $k$th power of a graph ($k \geq 3$) leaves a Hamiltonian graph, and that removing as many as $k-3$ vertices from the $k$th power of a graph ($k \geq 3$) leaves a Hamiltonian connected graph. Further, if every vertex of a graph has degree two or more, then the square of the graph contains a 2-factor. Finally, we show that the squares of certain Euler graphs are Hamiltonian.

Key words: Combinatorics; connectivity; Euler graphs; 2-factors; graph theory; Hamiltonian circuits; Hamiltonian connected; powers of graphs.

## 1. Introduction

We use the notation and terminology of [11][1], with the terms "point", "line", and "cycle" replaced by *vertex*, *edge*, and *circuit*. Further, we denote the set of edges of a graph $G$ by $E(G)$. We follow the practice of representing a path by the sequence of vertices of the path. To distinguish between a path (circuit) $p$ and the graph whose vertices and edges are exactly those of $p$, we denote the graph by $|p|$, and we call the graph a *pathoid (circuitoid)*. We denote the distance between two vertices $\alpha$ and $\beta$ in a graph $G$ by $d_G(\alpha, \beta)$, and we denote the degree of a vertex $\alpha$ in $G$ by $v_G(\alpha)$. The undirected edge joining vertices $\alpha$ and $\beta$ is denoted by $(\alpha, \beta)$ or $(\beta, \alpha)$ interchangeably.

Given an integer $k \geq 1$, the $k$th *power* $G^k$ of a graph $G$ is a graph with $V(G^k) = V(G)$ and $(\alpha, \beta) \in E(G^k)$ iff $d_G(\alpha, \beta) \in \{1, 2, \ldots, k\}$. $G^2$ is called the *square* of $G$, and $G^3$ is called the *cube* of $G$. Given a path $p = \alpha_0, \alpha_1, \ldots, \alpha_k$, we let $F(p) = \alpha_0$, $L(p) = \alpha_k$, $V(p) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$, and $I(p) = V(p) - \{F(p), L(p)\}$. We call $p$ an $[\alpha_0, \alpha_k]$-*path* iff $F(p) = \alpha_0$ and $L(p) = \alpha_k$. Paths $p$ and $q$ are *internally disjoint* iff $I(p) \cap I(q)$ is empty. The number of elements in a set $S$ is denoted by $|S|$.

## 2. Connectivity

Our first two theorems give useful information about a property of raising a graph to a $k$th power and the structure of a graph once the operation has been carried out. The first theorem is an easy consequence of the definition of power of a graph.

THEOREM 1: *Let* G *be a graph and let* k = mn, *where* m *and* n *are both positive integers. Then* $G^k = (G^m)^n$.

We will use this theorem to show that the next theorem is best possible. The *connectivity* $\kappa(G)$ of a graph $G$ is the minimum over all pairs $\alpha, \beta$ of distinct vertices in $G$ of the maximum number of distinct internally disjoint $[\alpha, \beta]$-paths in $G$. Since raising a graph $G$ to the power $k$ usually increases the number of edges present, it is not unreasonable to conjecture that $G^k$ has higher connectivity than $G$ has. Thus,

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*AMS Subject Classification:* 0540.

*An invited paper. Partial support for this research has come to me through Professor W. T. Tutte from the National Research Council of Canada.

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[1] Figures in brackets indicate the literature references at the end of this paper.

THEOREM 2: *Let* G *be a finite graph, and let* k *be a positive integer. Then*

$$\kappa(G^k) \geq \min(|V(G)|-1, k\kappa(G)).$$

PROOF: If $\kappa(G)=0$ or $k=1$, the theorem is immediate. Similarly, if $G^k$ is complete, the theorem holds because in that case, $\kappa(G^k)=|V(G)|-1$.

Now suppose $G^k$ is not complete, $\kappa(G)>0$, and $k>1$. Let $\xi$ and $\eta$ be vertices of $G$ which are not adjacent in $G^k$. Then by the definition of $G^k$, $d_G(\xi, \eta)>k$. Let $a_1, a_2, \ldots, a_{\kappa(G)}$ be $\kappa(G)$ internally disjoint $[\xi, \eta]$-paths in $G$, with $a_i=\xi, \gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,r_i}, \eta$. Since $d_G(\xi, \eta)>k$, $r_i \geq k$ whenever $i \in \{1, 2, \ldots, \kappa(G)\}$. Suppose $r_i=kt_i+s_i$, with $0 \leq s_i \leq k-1$. Then in $G^k$, $\xi$ and $\eta$ are joined by the paths

$$\xi, \gamma_{i,1}, \gamma_{i,k+1}, \gamma_{i,2k+1}, \ldots, \gamma_{i,t_ik+1}, \eta=p_{i,1},$$

$$\xi, \gamma_{i,2}, \gamma_{i,k+2}, \gamma_{i,2k+2}, \ldots, \gamma_{i,t_ik+2}, \eta=p_{i,2},$$

$$\ldots$$

$$\xi, \gamma_{i,s_i}, \gamma_{i,k+s_i}, \gamma_{i,2k+s_i}, \ldots, \gamma_{i,t_ik+s_i}, \eta=p_{i,s_i},$$

$$\xi, \gamma_{i,s_i+1}, \gamma_{i,k+s_i+1}, \gamma_{i,2k+s_i+1}, \ldots, \gamma_{i,(t_i-1)k+s_i+1}, \eta=p_{i,s_i+1},$$

$$\xi, \gamma_{i,s_i+2}, \gamma_{i,k+s_i+2}, \gamma_{i,2k+s_i+2}, \ldots, \gamma_{i,(t_i-1)k+s_i+2}, \eta=p_{i,s_i+2},$$

$$\ldots$$

$$\xi, \gamma_{i,k}, \gamma_{i,2k}, \ldots, \gamma_{i,(t_i-1)k}, \eta=p_k.$$

Each of these $k$ paths contains only vertices of $a_i$ and each path $a_i$ contains vertices which can be used to form $k$ internally disjoint $[\xi, \eta]$-paths in $G^k$ in this way. Since the paths $a_1, \ldots, a_{\kappa(G)}$ are internally disjoint, between $\xi$ and $\eta$ in $G^k$ there are $k\kappa(G)$ internally disjoint $[\xi, \eta]$-paths. Thus, $G^k$ is $k\kappa(G)$-connected.

COROLLARY 2A: *If* G *is a finite graph with connectivity* $\kappa(G) \geq 1$, *if* k *is a positive integer, and if* $k\kappa(G) \geq |V(G)|-1$, *then* $G^k$ *is a complete graph.*

Theorem 2 is best possible in the sense that additional conditions are needed to improve it, as the following examples show. If $P$ is a pathoid with $n$ vertices, if $\xi \in V(P)$ and has degree 1, and if $r$ is a positive integer less than $n$, then $\xi$ is of degree $r$ in $P^r$. Thus, by Theorem 2, $P^r$ has connectivity exactly $r$. Given positive integers $k$ and $m$, and given $n \geq km+1$, let $P$ be a pathoid of length $n-1$. Then $(P^m)^k=P^{mk}$ by Theorem 1, and so it is a graph of connectivity $km$ which is the $k$th power of an $m$-connected graph. Finally, let $C_n$ be a circuitoid with $n$ vertices. Let $m>0$ and even, and let $k$ be a positive integer and $n$ be an integer no less than $km+1$; then $C_n^{(km)/2} = (C_n^{m/2})^k$ and is a minimally $km$-connected graph which is the $k$th power of a minimally $m$-connected graph.

## 3. *r*-Hamiltonian Powers of Graphs

Following [4] and [17], we call a graph $G$ *Hamiltonian connected* iff every two distinct vertices of $G$ are joined by a Hamiltonian path in $G$, and we call $G$ *r-Hamiltonian* iff $|V(G)| \geq r+3$ and $G-\{\xi_1, \ldots, \xi_r\}$ is Hamiltonian for every set of $r$ vertices $\{\xi_1, \ldots, \xi_r\} \subseteq V(G)$. Given a path $p$, we let $l(p)$ denote the length of $p$. Given disjoint paths $p=\xi_1, \xi_2, \ldots, \xi_r$ and $q=\eta_1, \eta_2, \ldots, \eta_s$, if $\xi_r$ is adjacent in $G$ to $\eta_1$, we denote the path $\xi_1, \ldots, \xi_r, \eta_1, \eta_2, \ldots, \eta_s$ by $(p), (q)$. Let $\xi$ be a cut

vertex of a connected graph $G$, and let $H_1, \ldots, H_r$ be all of the components of $G - \xi$. Then for each $j \in \{1, \ldots, r\}$, we call the subgraph

$$G - \bigcup_{\substack{i=1 \\ i \neq j}}^{r} V(H_i)$$

of $G$ a *$\xi$-section* of $G$.

CONJECTURE 1: *If* G *is a finite connected graph,* k *is an integer no less than 3, and* $A \subseteq V(G)$ *such that* $|A| \leq k\kappa(G) - 3$, *then* $G^k - A$ *is Hamiltonian connected.*

CONJECTURE 2: *If* G *is a finite connected graph,* k *is an integer no less than 3, and* $r \leq min(|V(G)| - 3, k\kappa(G) - 2)$, *then* $G^k$ *is* r*-Hamiltonian.*

Because Theorem 2 is best possible and at least 2-connectedness is required for the presence of a Hamiltonian circuit in a graph, Conjecture 2 is best possible if it is true. Furthermore, if $0 \leq |V(G)| - 3 \leq k\kappa(G) - 2$, then $|V(G)| - 1 \leq k\kappa(G)$ and $G^k$ is complete by Corollary 2A. In such a case, for any set $A \subseteq V(G)$ with $|A| \leq |V(G)| - 3$, $G^k - A$ is Hamiltonian. Thus we need only consider Conjecture 2 for the case in which $|V(G)| - 3 \geq k\kappa(G) - 2$ and $|A| \leq k\kappa(G) - 2$. We have not yet proven these conjectures in general, but in the next portion of this paper we give proofs for the case of $\kappa(G) = 1$.

Let $G$ be a graph, $k$ a positive integer, and $A$ a subset of the vertices of $G$. Two distinct vertices $\xi$ and $\eta$ of $G$ are *$A$-joined* iff there exists a path $p$ in $G$ joining $\xi$ and $\eta$ such that $I(p) \subseteq A$. Note that two adjacent vertices are *$A$-joined* for every set $A$ of vertices in $G$. A path $p$ of $G$ is *$(A, k)$-solid* iff $\{F(p), L(p)\} \subseteq V(G) - A$, $|I(p) - A| \leq 2$, and $l(p) > k$.

LEMMA 1: *Let* $k \geq 3$ *be an integer,* G *a finite graph, and* $A \subseteq V(G)$ *with* $|A| \leq k - 3$. *Then* G *contains no* (A, k)*-solid paths.*

PROOF: This lemma is immediate from the definition of $(A, k)$-solid paths.

LEMMA 2: *Let* $k \geq 3$ *be an integer, let* G *be a graph, and let* $A \subseteq V(G)$ *such that* $|A| \leq k - 2$. *If* a *is a path of* G *such that* $|I(a) - A| \leq 1$, *then* $l(a) \leq k$.

PROOF: Since $|A| \leq k - 2$ and $|I(a) - A| \leq 1$, $|I(a)| \leq k - 1$. But $l(a) = |I(a)| + 1 \leq k - 1 + 1 = k$.

We next prove a very strong result that holds for special subsets $A$ of vertices of a tree. Let $\phi$ be the empty set.

LEMMA 3: *Let* $k \geq 3$ *be an integer, let* T *be a finite tree, and let* $A \subseteq V(T)$. *Suppose no path in* T *is* (A, k)*-solid. If* $\xi$ *and* $\eta$ *are distinct* A*-joined vertices in* $V(T) - A$, *then* $T^k - A$ *contains a Hamiltonian* $[\xi, \eta]$*-path.*

PROOF: If $|V(T)| \leq k + 1$, then $T^k$ is complete by Corollary 2A. Hence $T^k - A$ is a complete graph with at least the two vertices $\xi$ and $\eta$, and so there is a Hamiltonian $[\xi, \eta]$-path in $T^k - A$.

Suppose the lemma is true for all trees $T'$ with $|V(T')| < q$, and let $T$ be a tree with $q$ vertices which satisfies the conditions of the lemma. Let $T_1, T_2, \ldots, T_r$, $r \geq 1$, be the $\xi$-sections of $T$ for which $V(T_i) - A \neq \phi$, and suppose $\eta \in V(T_r)$. Since a path which is $(A, k)$-solid in $T_i$, $1 \leq i \leq r$, is $(A, k)$-solid in $T$, the lemma holds for all subtrees $T_i$ which have at least 2 vertices not in $A$. In tree $T_i$, $1 \leq i \leq r - 1$, let $\gamma_{i,1}$ be a vertex of $V(T_i) - A$ which is $A$-joined to $\xi$ in $T$, and let $\gamma_{r,1} = \eta$, which is $A$-joined to $\xi$ by assumption. In tree $T_i$, $1 \leq i \leq r$, if $V(T_i) - (A \cup \{\gamma_{i,1}\}) \neq \phi$, let $\gamma_{i,2}$ be a vertex of $V(T_i) - (A \cup \{\gamma_{i,1}\})$ which is $A$-joined to $\gamma_{i,1}$ in $T$, and if $V(T_i) - (A \cup \{\gamma_{i,1}\}) = \phi$, let $\gamma_{i,2} = \gamma_{i,1}$.

Since $T$ is a tree, there is a unique $[\xi, \gamma_{1,2}]$-path $a$ in $T$, and $\gamma_{1,1}$ is the only vertex in $I(a) - A$; by the definition of $(A, k)$-solid path, $\ell(a) \leq k$ since $a$ is not $(A, k)$-solid. Thus $d_T(\xi, \gamma_{1,2}) \leq k$. Further, at most two vertices, $\xi$ and $\gamma_{i,1}$, are in $I(s_i) - A$, where $s_i$ is the unique $[\gamma_{i-1,1}, \gamma_{i,2}]$-path in $T$, $2 \leq i \leq r$; since $T$ contains no $(A, k)$-solid path, $s_i$ cannot have length greater than $k$, so $d_T(\gamma_{i-1,1}, \gamma_{i,2}) \leq k$ for $i \in \{2, \ldots, r\}$. Since $T_1, \ldots, T_r$ satisfy the conditions of this lemma and have fewer than $q$ vertices each, in $T_i^k - A$ there is a Hamiltonian $[\gamma_{i,2}, \gamma_{i,1}]$-path $a_i$. Thus, $\xi, (a_1), (a_2), \ldots, (a_r)$ is a Hamiltonian $[\xi, \eta]$-path in $T^k - A$.

Examples of trees $T$ and sets $A$ which satisfy the conditions of Lemma 3 are shown in figure 1, where the vertices in the sets A are contained in dashed curves.

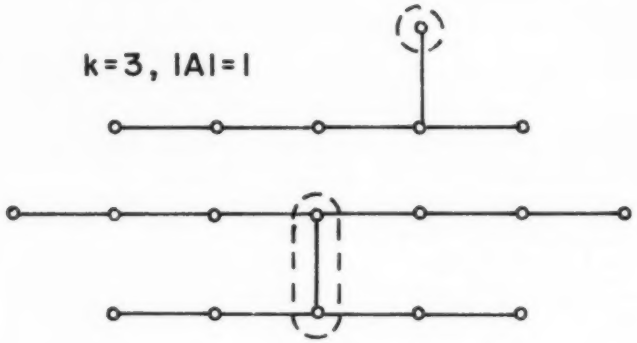


FIGURE 1

COROLLARY 3a: *Let* T *be a finite tree with at least 3 vertices, let* k *be an integer no less than 3, and let* $A \subseteq V(T)$ *such that no path of* T *is* (A, k)*-solid. Then, for each edge* $\lambda \in E(T-A)$, $T^k - A$ *contains a Hamiltonian circuit* c *with* $\lambda \in E(c)$.

THEOREM 3: *Let* T *be a finite tree, let* k *be an integer no less than* 3, *and let* $A \subseteq V(T)$. *Suppose* T *has no* (A, k)*-solid paths. Then* $T^k - A$ *is Hamiltonian connected.*

PROOF: The theorem is clearly true for a tree with 1 or 2 vertices, and it is vacuously true for a tree with no vertices. Suppose the theorem is true for every tree with fewer than $q$ vertices, and let $T$ be a tree with $q$ vertices. By Lemma 3, we need only show that, for any two distinct vertices which are not $A$-joined in $T$, $T^k - A$ contains a Hamiltonian path joining them. Let $\xi$ and $\eta$ be two vertices of $V(T) - A$ which are not $A$-joined. Let $a$ be the $[\xi, \eta]$-path in $T$ and let $\alpha \in I(a) - A$. Let the components of $T - \alpha$ which contain vertices not in $A$ be $C_1, \ldots, C_r$, with $\xi \in V(C_1)$ and $\eta \in V(C_r)$.

For $i \in \{2, 3, \ldots, r-1\}$, let $\rho_i$ be a vertex of $C_i - A$ which is $A$-joined to $\alpha$ in $T$. Let $\rho_1 = \xi$. If there is a vertex of $C_r - A$ other than $\eta$ which is $A$-joined to $\alpha$, let $\rho_r$ be such a vertex, and if $\eta$ is the only vertex in $C_r - A$, let $\rho_r = \eta$. Otherwise, let $\rho_r$ be a vertex of $C_r - A$ which is $A$-joined to $\eta$. For $i \in \{2, 3, \ldots, r-1\}$, let $\delta_i$ be a vertex of $C_i - A$ other than $\rho_i$ which is $A$-joined to $\rho_i$ (if no such vertex exists, let $\delta_i = \rho_i$). If there is a vertex of $C_1 - A$ other than $\xi$ which is $A$-joined to $\alpha$ in $T$, let $\delta_1$ be such a vertex, and if $\xi$ is the only vertex in $C_1 - A$, let $\delta_1 = \xi$. Otherwise, let $\delta_i$ be a vertex which is $A$-joined to $\xi$. Let $\delta_r = \eta$. By these choices, $\xi$ is the only vertex not in $A$ which can be on the path of $T$ joining $\delta_1$ and $\alpha$, and $\eta$ is the only vertex not in $A$ which can be on the path of $T$ joining $\alpha$ and $\rho_r$.

Each component $C_i$ is a tree with fewer than $q$ vertices. Further, since $T$ contains no $(A, k)$-solid paths, $C_i$ contains no $(A, k)$-solid paths, for each $i$. Thus the theorem holds for each tree $C_i$, and $C_i^k - A$ contains a Hamiltonian $[\rho_i, \delta_i]$-path $p_i$.

We note that in a tree $S$ with no $(A, k)$-solid paths, if $p$ is a path for which $\{F(p), L(p)\} \subseteq V(S) - A$ and $|I(p) - A| \leq 2$, then $l(p) \leq k$. Hence, $d_T(\delta_1, \rho_2) \leq k$ since at most $\xi$ and $\alpha$ of $V(T) - A$ can be on the $[\delta_1, \rho_2]$-path of $T$. Also, $d_T(\delta_i, \rho_{i+1}) \leq k$ whenever $i \in \{2, 3, \ldots, r-2\}$, since at most $\rho_i$ and $\alpha$ of $V(T) - A$ can be on the $[\delta_i, \rho_{i+1}]$-path in $T$. Further, $d_T(\delta_{r-1}, \alpha) \leq k$ since $\rho_{r-1}$ is the only vertex not in $A$ which can be on the $[\delta_{r-1}, \alpha]$-path of $T$, and $d_T(\alpha, \rho_r) \leq k$ since $\eta$ is the only vertex not in $A$ which can be on the $[\alpha, \rho_r]$-path in $T$. Therefore, $(p_1), (p_2), \ldots, (p_{r-1}), \alpha, (p_r)$ is a Hamiltonian $[\xi, \eta]$-path in $T^k - A$.

COROLLARY 3A: *Let* G *be a connected finite graph, let* k *be an integer no less than 3, and let* $A \subseteq V(G)$ *with* $|A| \leq k-3$. *Then* $G^k - A$ *is Hamiltonian connected.*

PROOF: Let $T$ be a spanning tree in $G$. Since $|A| \leq k-3$, $T$ contains no $(A, k)$-solid paths by Lemma 1. Therefore, $T^k - A$ is Hamiltonian connected by Theorem 3. But $T^k - A$ is a spanning subgraph of $G^k - A$; thus $G^k - A$ is Hamiltonian connected.

Corollary 3A proves Conjecture 1 for the case of $\kappa(G) = 1$. We can improve this result slightly in the direction of Conjecture 1 as follows:

COROLLARY 3B: *Let* G *be a connected finite graph and let* k *be an integer no less than 3. Let* $A \subseteq V(G)$ with $|A| \leq k + \kappa(G) - 4$. *Then* $G^k - A$ *is Hamiltonian connected.*

PROOF: $A$ can be expressed as the union of disjoint sets $B$ and $C$ such that $|B| \leq k-3$ and $|C| \leq \kappa(G) - 1$. From the definition of $\kappa(G)$, $G - C$ is connected; thus $(G-C)^k - B$ is Hamiltonian connected by Corollary 3A. But $(G-C)^k$ is a spanning subgraph of $G^k - C$ and $B \cup C = A$; hence $(G-C)^k - B$ is a spanning subgraph of $G^k - A$, and $G^k - A$ is Hamiltonian connected.

LEMMA 4: *Let* k *be an integer no less than 2, let* G *be a graph, and let* $A \subseteq V(G)$ *such that* $|A| \leq k-2$. *Let* p *be an* (A, k)*-solid path in* G. *Then* $A \subseteq I(p)$.

PROOF: By definition, $l(p) = |I(p)| + 1$ and $|I(p) - A| \leq 2$. Therefore, $k < l(p) = |I(p)| + 1 = |I(p) \cap A| + |I(p) - A| + 1 \leq |I(p) \cap A| + 3$, or $k - 2 \leq |I(p) \cap A|$. But $|A| \leq k-2$. Thus, $A \subseteq I(p)$.

In a paper published in 1960, Sekanina [18] proved

THEOREM (Sekanina's Theorem): *If* G *is a connected finite graph, then* $G^3$ *is Hamiltonian connected.*

This theorem was proved again by Karaganis in a paper [15] published in 1968. Using this result, Chartrand and Kapoor [2] proved

THEOREM A: *If* G *is a connected finite graph with at least 4 vertices, then* $G^3$ *is 1-Hamiltonian.*

A proof by construction for the next theorem was recently published in [1]. However, the following proof is belived to have points of sufficient interest to warrant its publication. Recall that if $p$ is a path, then $|p|$ is the graph whose edges and vertices are precisely those of $p$.

THEOREM 4: *Let* k *be an integer no less than 3, and let* G *be a connected finite graph with at least* k+1 *vertices. Then* $G^k$ *is* r*-Hamiltonian for every integer* r *in* $\{0, 1, \ldots, k-2\}$.

PROOF: Let $A \subseteq V(G)$ with $|A| \leq k-2$. If $|A| \leq 1$, then $G^3 - A$ contains a Hamiltonian circuit by Theorem A. Since $G^3 - A$ is a spanning subgraph of $G^k - A$, $G^k - A$ is Hamiltonian.

Suppose $|A| \geq 2$. Since $G$ is connected, we may choose a spanning tree $T$ of $G$. If $T$ has no $(A, k)$-solid paths, then $T^k - A$ is Hamiltonian connected by Theorem 3, and so it contains a Hamiltonian circuit. Since $T^k - A$ is a spanning subgraph of $G^k - A$, $G^k - A$ is Hamiltonian.

Now we assume that $T$ contains an $(A, k)$-solid path $a$; by Lemma 4, $A \subseteq I(a)$. Since $|A| \geq 2$, there is an edge $\lambda = (\xi_1, \xi_2)$ in $E(a)$ such that each component of $|a| - \lambda$ contains one or more vertices of $A$. Let $T_i$ be the component of $T - \lambda$ which contains $\xi_i$, $i \in \{1, 2\}$. For $i \in \{1, 2\}$, if $\xi_i \notin A$, we let $\rho_i = \xi_i$, and if $\xi_i \in A$, we let $\rho_i$ be a vertex of $T_i$ which is not in $A$ and which is $A$-joined to $\xi_i$ (such a vertex exists because the ends of $a$ are not in $A$). Let $\delta_i$ be a vertex of $V(T_i) - A$ different from $\rho_i$ which is $A$-joined to $\rho_i$ (if no such vertex exists, let $\delta_i = \rho_i$). Since both of $T_1$ and $T_2$ contain vertices of $A$, $|V(T_i) \cap A| < k-2$ for each $i$; hence $T_i$ satisfies the conditions of Lemma 1 and contains no $(A, k)$-solid arc. Then applying Lemma 3 to $T_i$, $i \in \{1, 2\}$, let $a_i$ be a Hamiltonian path in $T_i^k - A$ from $\rho_i$ to $\delta_i$ (if $\rho_i = \delta_i$, let $a_i = \rho_i$). Since $\rho_1$ is the only vertex of $V(T) - A$ which can lie on the $[\delta_1, \rho_2]$-path of $T$, $d_T(\delta_1, \rho_2) \leq k$ by Lemma 2. Similarly, $d_T(\delta_2, \rho_1) \leq k$. Hence $(a_1)$, $(a_2)$, $\rho_1$ is a Hamiltonian circuit in $T^k - A$, and it is also a Hamiltonian circuit in $G^k - A$.

The following example shows that Theorem 4 cannot be strengthened to the level of Theorem 3, and Corollary 3A is best possible in the usual sense. In the tree $T$ shown in figure 2A, the five vertices surrounded by the dashed curve are the vertices in $A$, and $k = 7$. $T^7 - A$ is shown in figure 2B. In $T$, $\alpha$ and $\beta$ are $A$-joined, but it is easily shown that in $T^7 - A$, there is no Hamiltonian $[\alpha, \beta]$-path ($\gamma$ would have to be both the successor of $\alpha$ and the predecessor of $\beta$ in any such path).

The following corollary provides a slight weakening of the conditions of Theorem 4, with a corresponding weakening of the conclusion:

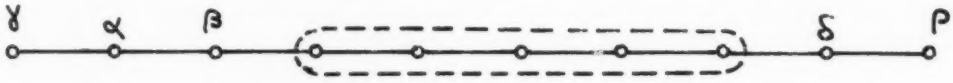


FIGURE 2A

FIGURE 2B

COROLLARY 4A: *Let* G *be a connected finite graph with at least 3 vertices, let* k *be an integer no less than 3, and let* A *be a subset of* V(G) *such that* $|A| \leq min\ (k-1, |V(G)|-2)$. *Then* $G^k - A$ *contains a Hamiltonian path.*

PROOF: We simply remove all but one of the vertices of $A$ from $G^k$. By Theorem 4, the resulting graph contains a Hamiltonian circuit. Removing one more vertex converts the circuit into a Hamiltonian path in the resulting graph, which is $G^k - A$.

While we have not proved Conjecture 2, Theorem 4 allows us to come a little closer to the statement of the conjecture, as the following corollary shows:

COROLLARY 4B: *Let* k *be an integer no less than 3 and let* G *be a connected finite graph with at least* $k + \kappa(G)$ *vertices. Then* $G^k$ *is* r*-Hamiltonian for every* r *in* $\{0, 1, \ldots, k + \kappa(G) - 3\}$.

The proof of this corollary is similar to the proof of Corollary 3B.

Conjecture 2 might be extended to include the following:

CONJECTURE 3: *Let* G *be a graph with* $\kappa(G) \geq 2$ *and with at least* $2\kappa(G) + 1$ *vertices. Then* $G^2$ *is* $(2\kappa(G) - 2)$*-Hamiltonian.*

I suspect that Conjecture 3 does not hold (although it is valid if $G$ is a circuitoid). However, the following weaker theorem was recently proved [3, 14]:

THEOREM: *Let* G *be a graph with* $\kappa(G) \geq 2$ *and with at least 4 vertices. Then* $G^2$ *is 1-Hamiltonian.*

## 4. Squares of Graphs

The set of all vertices of $G$ having degree $k$ is denoted by $V_k(G)$. A *caterpillar* is a tree $T$ such that $T - V_1(T)$ is a pathoid or the empty graph. A *2-factor* of a graph $G$ is a subgraph $H$ of $G$ such that $V(H) = V(G)$ and every vertex in $H$ has degree 2.

In contrast to the situation for higher powers of graphs, we do not yet know which graphs have Hamiltonian squares. Neuman [16] has characterized those trees for which a Hamiltonian path joining two specified vertices will exist in the square of the trees. An easy consequence of his characterization is the following result:

THEOREM 5: *The square of a tree is Hamiltonian if and only if the tree is a caterpillar with at least 3 vertices.*[2]

H. Fleischner [6, 7, 8, 9] has proved that the square of a block is Hamiltonian, and he has characterized those cubic graphs with Hamiltonian squares [5, 10]. A few other more special results are known. The remainder of this paper contains two further theorems connected with the problem of determining which graphs have Hamiltonian squares.

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[2] Neuman's result can also easily be used to show that the square of a tree is Hamiltonian connected if and only if the tree is a caterpillar with at most one vertex of degree greater than one.

If the square of a graph contains a Hamiltonian circuit, it certainly contains a 2-factor. Thus, it is reasonable to consider the question of which squares of graphs contain 2-factors ($S(K_{1,3})$ is an example of a graph whose square does *not* contain a 2-factor). In this connection, the following theorem is of interest. Given a path $p$, the *internal vertices* of $p$ are the vertices in $I(p)$. We denote the degree in $G$ of a vertex $\xi$ by $v_G(\xi)$.

THEOREM 6: *Let* G *be a graph with minimum degree at least 2. Then* $G^2$ *contains a 2-factor.*

PROOF: By Theorem 5, it is sufficient to find a spanning forest in $G$ in which each tree is a caterpillar with at least 3 vertices. We note first that since $G$ has minimum degree at least 2, each component of $G$ contains at least 3 vertices, and a longest path in $G$ must have length at least 2.

Choose a longest path $a_1$ in $G$. Having chosen paths $a_1, \ldots, a_k$ in $G$, if $H_k = G - \bigcup_{i=1}^{k} V(a_i)$ is not empty, choose a longest path $a_{k+1}$ in $H_k$. Note that the end vertices of $a_{k+1}$ can be adjacent only to vertices in $\bigcup_{i=1}^{k} I(a_i) \cup V(a_{k+1})$. Eventually, since $G$ is finite, we find we have chosen paths $a_1, \ldots, a_r$ such that $H_r$ is the empty graph. Now for each end vertex $\mu$ of each path of length 0 or 1, choose an internal vertex of a longer path to which $\mu$ is adjacent through an edge $\lambda_\mu$ (such an internal vertex must exist since $v_G(\xi) \geq 2$ for all $\xi \in V(G)$). We form a spanning forest whose trees are caterpillars by deleting the edges from those pathoids in $\{/a_1/, \ldots, /a_r/\}$ which have length 1 and adding the edges $\lambda_\mu$ to form the caterpillars. Since each of these caterpillars includes the vertices of a path with an internal vertex, each caterpillar has at least three vertices.

An *Euler* graph is a graph in which every vertex has even degree. In a path $p = (r), \xi, (s)$, the *neighbors* of the vertex $\xi$ are the vertices $L(r)$ and $F(s)$. Given a path $p = \xi_1, \xi_2, \ldots, \xi_{r-1}, \xi_r$, we let $p^{-1}$ denote the path $\xi_r, \xi_{r-1}, \ldots, \xi_2, \xi_1$. We call a path $x$ a *section* of a path $p$ iff there exist paths $y$ and $z$ such that $p = (y), (x), (z)$ ($y$ or $z$ may be empty). Recall that all walks in this paper are denoted by sequences of vertices.

The next theorem is a beginning for the study of the question of which Euler graphs have Hamiltonian squares. Fleischner's proof that the square of a block is Hamiltonian involves finding a Hamiltonian circuit in the square of an Euler graph (see especially [7]); thus, a thorough understanding of Euler graphs with Hamiltonian squares would be quite desirable. The following Theorem 7, whose statement is quite long, can be summarized as follows: "If $G$ is an Euler graph with at least 3 vertices such that $G - V_2(G)$ is a forest, then $G^2$ is Hamiltonian. Further, $G^2$ contains a Hamiltonian circuit which includes many of the edges of $G$."

THEOREM 7: *Let* G *be a connected Euler graph with at least 3 vertices such that* $G - V_2(G)$ *is a forest with trees* $T_1, \ldots, T_r$, $r \geq 0$. *Let* e *be an Euler trail in* G. *In each tree* $T_i$, $i \in \{1, 2, \ldots, r\}$, *choose one vertex* $\eta_i$ *and call it the root of* $T_i$. *Then* $G^2$ *contains a Hamiltonian circuit* h *such that:*

(1) h *is a subsequence of* e, *where both* h *and* e *are denoted by a sequence of vertices;*

(2) *the root* $\eta_i$ *of each tree* $T_i$ *is adjacent in* G *to its two neighbors in* h; *and*

(3) *for each path* s *in* G *such that* $V(s) \subseteq V_2(G)$, *either* s *or* $s^{-1}$ *is a section of* h.

PROOF: The theorem is trivial if $G$ is a circuitoid. Thus we may assume $r \geq 1$. Define a function $\Psi$, with domain $V(G) - (V_2(G) \cup \{\eta_1, \eta_2, \ldots, \eta_r\})$, as follows: for $\xi \neq \eta_i$ in $V(T_i)$, let $\Psi(\xi)$ be the second vertex on the unique $[\xi, \eta_i]$-path in $T_i$. In $e$, mark each vertex of $V_2(G)$, and mark one occurrence of each of $\eta_1, \ldots, \eta_r$. Also for each vertex $\xi \in V(G) - (V_2(G) \cup \{\eta_1, \ldots, \eta_r\})$, mark precisely that occurrence of $\xi$ in $e$ for which $\Psi(\xi)$ is a neighbor of $\xi$ at that location in $e$. Form $h$ from $e$ by deleting all unmarked vertices in $e$. This choice of $h$ satisfies condition (1) of the theorem.

Since we left in $h$ one occurrence of each of the distinct vertices in $e$, and since $V(e) = V(G)$, $V(h) = V(G)$. Note that the unmarked vertices of $e$ are all in the forest $G - V_2(G)$. Two vertices of $G - V_2(G)$ which are successive in $e$ and are adjacent in that subgraph are in the same tree $T_i$. Hence one is the second vertex in the path joining the other to $\eta_i$, so that one of them is necessarily marked at that part of $e$. Thus no two successive vertices in $e$ are both unmarked, and for every section $\omega, \chi$ of $h$, there is at most one vertex between $\omega$ and $\chi$ in $e$ at the corresponding location in $e$. Since $e$ is a trail in $G$, $d_G(\omega, \chi) \leq 2$. Thus, $h$ is a Hamiltonian circuit in $G^2$.

For each $i \in \{1, 2, \ldots, r\}$, since $\eta_i$ is $\Psi(\xi)$ for every vertex $\xi$ adjacent to $\eta_i$ in $T_i$, and since no vertex of degree 2 in $G$ is deleted in obtaining $h$ from $e$, no neighbors of $\eta_i$ in $e$ are deleted when forming $h$. Thus, in the one occurrence of $\eta_i$ in $h$, both neighbors of $\eta_i$ in $h$ are neighbors of $\eta_i$ in $e$ and hence are adjacent to $\eta_i$ in $G$. Finally, for each path $s$ in $G$ such that $V(s) \subseteq V_2(G)$, either $s$ or $s^{-1}$ is a section of $e$. Since no vertices of degree 2 in $G$ are deleted in forming $h$, $s$ or $s^{-1}$ remains a section of $h$. Thus $h$ satisfies conditions (2) and (3) of the theorem.

For any $\xi \in V(G)$, if $\xi$ and $\eta_i$ are adjacent in $G$ they must be neighbors at some occurrence of $\eta_i$ in $e$. Then at that occurrence of $\eta_i$, $\eta_i$ can be marked. Thus, for each root $\eta_i$, we may select one vertex adjacent to $\eta_i$ in $G$ which will be a neighbor of $\eta_i$ in $h$.

Let $G$ be a graph whose square is Hamiltonian, let $h$ be a Hamiltonian circuit in $G$, and let $\lambda$ be an edge of $G$ in $h$. Let $\xi$ be one of the ends of $\lambda$. Further, let $H_1$ and $H_2$ be disjoint caterpillars disjoint from $G$, and let $\eta_i$ be a vertex of degree no more than 1 in $H_i - V_1(H_i)$, for $i \in \{1, 2\}$ (if $H_i$ has only two vertices, let $\eta_i$ be one of them). Form a graph $M$ from $H_1$ and $G$ by identifying $\eta_1$ and $\xi$. Then $M^2$ is Hamiltonian as indicated by figure 3, where the dashed line indicates the sequence inserted between the ends of $\lambda$ in $h$, (in this and following figures, a wavy line is used to denote $h$). Variations in the length of $H_1 - V_1(H_1)$ (particularly variations to odd lengths) and changes in the number of vertices of degree 1 in $H_1$ can be dealt with in an obvious manner. Note that one edge of $H_1$ is included in the Hamiltonian circuit of $M^2$.

Now let $H_1$ and $H_2$ be as before, but suppose $G^2$ contains a Hamiltonian circuit $h$ which includes two edges $\lambda$ and $\mu$ of $G$, both incident with the same vertex $\xi$ of $G$. In this case, let $N$ be formed from $H_1$, $H_2$, and $G$ by identifying $\eta_1$, $\eta_2$, and $\xi$. Then $N^2$ contains a Hamiltonian circuit as indicated by figure 4, where the dashed lines indicate the sequence which is to replace $\xi$ in $h$. Variations in $H_1$ and $H_2$ are again easily dealt with. Note that an edge of each of $H_1$ and $H_2$ is in the Hamiltonian circuit of $N^2$.

Now let $G$ and $H$ be disjoint graphs such that $G^2$ and $H^2$ are both Hamiltonian. Let $\lambda$ be an edge of $G$ incident with a vertex $\xi$ and in a Hamiltonian circuit $g$ of $G^2$, and let $\mu$ be an edge of $H$ incident with a vertex $\eta$ and in a Hamiltonian circuit $h$ of $H^2$. Form a graph $P$ from $G$ and $H$ by identifying $\xi$ and $\eta$. Then $P^2$ contains a Hamiltonian circuit as indicated by the wavy and dashed curves in figure 5.

Theorem 7 describes many edges of $G$ which are included in a Hamiltonian circuit of $G^2$, where $G$ is an Euler graph in which $G - V_2(G)$ is a forest. In view of the techniques described in the preceding three paragraphs, Theorem 7 can thus be used to show many other graphs have Hamiltonian squares.

An Euler graph $G$ whose square is not Hamiltonian is shown in figure 6. It is easy to see that any Hamiltonian circuitoid in $G^2$ must contain edges joining a vertex other than a cut vertex of $G$ of each of the small triangles $T_1$, $T_2$, $T_3$, $T_4$, and $T_5$ to one of $\alpha$ or $\beta$. Thus one of $\alpha$ or $\beta$ must meet three edges

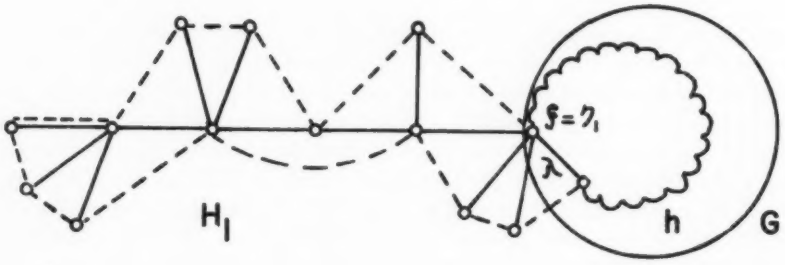


FIGURE 3

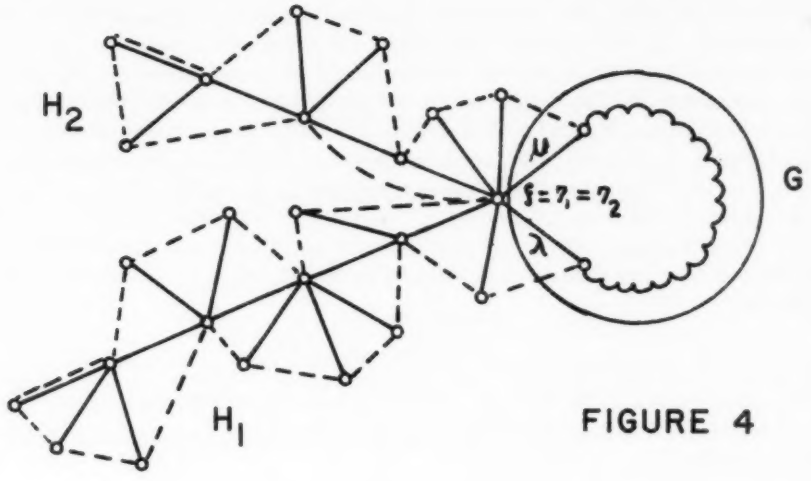


FIGURE 4

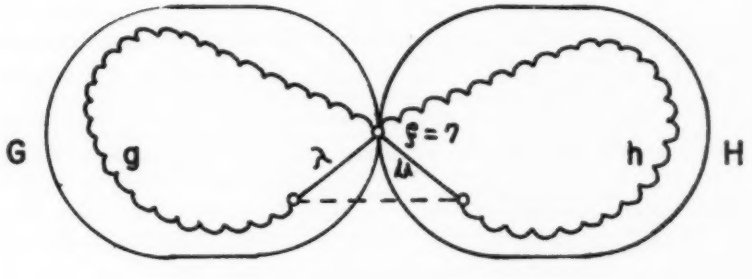


FIGURE 5

of the Hamiltonian circuitoid, which is impossible. I suspect that the graph of figure 6 is the smallest Euler graph whose square is not Hamiltonian (size being measured in number of vertices or number of edges).

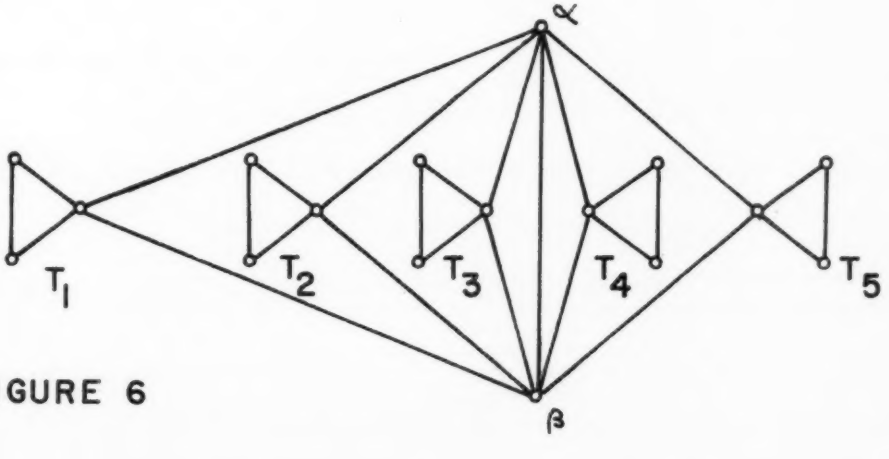


FIGURE 6

*ADDED IN PROOF:* Since this paper was submitted, references [6] and [7] have been combined into Fleischner, H., On spanning subgraphs of a connected bridgeless graph and their application to *DT*-graphs, J. Combinatorial Theory (to appear), and references [8] and [9] were combined into Fleischner, H., The square of every two-connected graph is Hamiltonian, J. Combinatorial Theory (to appear). Further, references [3] and [14] were combined into Chartrand, G., Hobbs, A. M., Jung, H. A., Kapoor, S. F., and Nash-Williams, C. St. J. A., The square of a block is Hamiltonian connected, J. Combinatorial Theory (to appear). It should be mentioned that the term "$\xi$-section" is similar to the term "$J$-component" of Tutte, W. T., The Connectivity of Graphs (Toronto Univ. Press, Toronto, 1967), and that many of the other terms in this paper not found in [11] were derived from the definitions in Nash-Williams, C. St. J. A., Graph-Theoretic Definitions (unpublished).

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My thanks go to Professors W. T. Tutte and C. St. J. A. Nash-Williams for their many useful suggestions. In particular, I wish to thank Professor Nash-Williams for suggesting the present form of the proof of Theorem 7.

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(Paper 77B1 & 2–374)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 1 & 2, January–June 1973

# On a Functional Equation*

**A. O. L. Atkin****

(November 16, 1972)

Subject to suitable conditions of continuity and normalization, it is shown that the equation $f(x, y)+f(x+y, z)=f(y, z)+f(x, y+z)$ implies $f(x, y)=g(x)+g(y)-g(x+y)$. The result has application in physics to the motion of an electron in a crystal lattice.

Key words: Analysis; continuous; equation; function; real.

The object of this note is to prove Lemma 1. The motivation is given in Lomont and Moses;[1] I am grateful to Professor Lomont for bringing this problem to my notice.

LEMMA 1: *Suppose that* $f:R\times R\to R$ *is continuous and satisfies*

$$f(x, y)+f(x+y, z)=f(y, z)+f(x, y+z) \tag{1}$$

*for all* x, y, z, *in* R. *Then there exists an unique continuous function* $g:R\to R$ *such that* $g(0)=g(1)$ *and* $f(x, y)=g(x)+g(y)-g(x+y)$ *for all* x, y, *in* R.

Without loss of generality we may assume $f(0, 0)=0$ (otherwise $f'=f-k, g'=g-k$, reduces). We use $(a, b, c)$ to denote the application of (1) with $x=a$, $y=b$, $z=c$. Then

$$\begin{aligned} &(x, 0, 0)\Rightarrow 2f(x, 0)=f(x, 0)\Rightarrow f(x, 0)=0 \quad &&\text{for all } x \text{ in } R.\\ &\text{Similarly} \qquad\qquad f(0, y)=0 \quad &&\text{for all } y \text{ in } R. \end{aligned} \tag{2}$$

We now prove

LEMMA 2: $f(x, y)=f(y, x)$ *for all* x, y, *in* R.

Define $F(x, y)=f(x, y)-f(y, x)$. Then $(x, y, x)\Rightarrow$

$$F(x, y)=F(x, x+y). \tag{3}$$

For nonnegative $N$ in $Z$ let $P(N)$ be the proposition:

"$F(mx, nx)=0$ for all $x$ in $R$ and all $m$, $n$, in $Z$ with $|m|\leq N$, $|n|\leq N$."

Then P(0). Assume $P(k-1)$ for a given $k\geq 1$. Then for $|n|\leq k-1$ we have by (3) that

$$F(kx, nx)=-F(nx, kx)=-F(nx, (k-n)x),$$

and the last term is zero either by hypothesis ($n\neq 0$) or by (2); replacing $x$ by $-x$ gives $F(-kx, nx)=F(nx, -kx)=0$. Finally $F(kx, kx)=F(kx, 0)=F(kx, -kx)$ by (3), and the middle term is zero by

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*AMS Subject Classification:* 3930.

*An invited paper.

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[1] J. S. Lomont and Harry E. Moses, The Application of Ray Representations of Translation Groups to the Motion of an Electron in a Crystal Lattice, Annals of Physics 67(2), 1971, pp. 406–431.

(2); replacing $x$ by $-x$ completes the proof of $P(k)$. Hence $P(N)$ for all $N$ by induction.

Thus $F(x, y)=0$ whenever $x:y$ is rational, and in particular when $x$ and $y$ are both rational. So $F$ is continuous on $R\times R$ and zero on a subset dense in $R\times R$, and hence identically zero. This proves Lemma 2.

*The construction of* g(x).

Denote by $S_N$ the set of all rational points $a/2^N$, with $a$ in $Z$ and $N$ nonnegative in $Z$. Then

$$S=\sum_{N=0}^{\infty} S_N \ . \ . \ . \supset S_N \supset \ . \ . \ . \supset S_1 \supset S_0.$$

We define a function $g: S\to R$ inductively as follows.

$$g(0)=g(1)=0;\ g(n)=g(n-1)-f(1, n-1)\ (n\geq 2);$$

$$g(n)=g(n+1)+f(1, n)\ (n\leq -1): \tag{4}$$

and for $N\geq 1$ and $x$ in $S_N-S_{N-1}$,

$$g(2^{-N})=g(2^{1-N})/2+f(2^{-N}, 2^{-N})/2;$$

$$g(x)=g(x-2^{-N})+g(2^{-N})-f(2^{-N}, x-2^{-N}). \tag{5}$$

LEMMA 3: *We have*

$$\mathrm{f(x, y)=g(x)+g(y)-g(x+y)}\ \textit{for all}\ \mathrm{x, y,}\ \textit{in}\ \mathrm{S.}$$

Define $F(x, y)=f(x, y)-g(x)-g(y)+g(x+y)$, so that $F(x, y)=F(y, x)$. Then by (4) $F(1, n)=0$ for all $n$ in $Z$, and $F(2^{-N}, x)=0$ for all $x$ in $S_{N-1}$ and for $x=2^{-N}$, if $N\geq 1$, by (5). Also $F$ clearly satisfies the functional equation (1). So for $x$, $y$, in $S_0(=Z)$ we have $(x, y, 1)\Rightarrow F(x, y)=F(x, y+1)$, whence $F(x, y)=0$ for all $x$, $y$ in $S_0$.

Assume now that $F(x, y)=0$ for all $x$, $y$ in $S_{k-1}$, for a given $k\geq 1$. Then if $x$ is in $S_k-S_{k-1}$, we have $z=x-2^{-k}$ in $S_{k-1}$, and $(2^{-k}, 2^{-k}, z)\Rightarrow F(2^{-k}, x)=0$. Using this, for $y$, $z$, in $S_k$ we have $(2^{-k}, y, z)\Rightarrow F(2^{-k}+y, z)=F(y, z)$, and these are now both equal to $F(2^{-k}+y, 2^{-k}+z)=F(y, 2^{-k}+z)$. One of the four equal terms has both arguments in $S_{k-1}$, and is zero by hypothesis. Thus we have shown that $F(y, z)=0$ for all $y$, $z$, in $S_k$. Lemma 3 now follows by induction.

LEMMA 4: Given $A>0$, $\epsilon>0$, there exists $\delta>0$ such that for all $x$, $y$, in $S$ with $|x-y|<\delta$ and $|x|\leq A$, $|y|\leq A$, we have $|g(x)-g(y)|<\epsilon$.

First $x$ in $S\Rightarrow 2x$ in $S$. Let $L=\limsup|g(x)|$ as $x\to 0$ in $S$. Then $g(2x)=2g(x)-f(x, x)$, so that $2L\leq L+\limsup_{x\to 0}|f(x, x)|=L$, and hence $L=0$ and $g(x)\to 0$ as $x\to 0$ in $S$. Now choose $\delta_1>0$ such that $|g(z)|<\epsilon/2$ for $|z|<\delta_1$, and $\delta_2>0$ such that $|f(x, z)|<\epsilon/2$ for $|z|<\delta_2$ and all $x$ satisfying $|x|\leq A$, $|z+x|\leq A$; the latter choice is possible by the uniform continuity of $f$ in a bounded region $R\times R$. Then with $x$, $z=y-x$, for $x$, $y$, in Lemma 3 we have for $|x-y|<\delta=\min(\delta_1, \delta_2)$

$$|g(x)-g(y)|<|f(x, z)|+|g(z)|<\epsilon,$$

as desired.

*Extension of* g(x) *to* R.

Given any $x$ in $R$, choose $x_n$ in $S$ such that $x_n\to x$ as $n\to\infty$, possible since $S$ is dense in $R$. As in the definition of real numbers using Cauchy sequences, Lemma 4 shows that $\lim_{n\to\infty} g(x_n)$ exists, and is independent of the choice of $x_n$, and we define $g(x)$ as this limit: $g(x)$ is also clearly continuous.

Finally we write $F(x,y)=f(x,y)-g(x)-g(y)+g(x+y)$ for $x,y$, in $R$. $F$ is continuous on $R\times R$, and by Lemma 3 $F$ is zero on $S\times S$ dense in $R\times R$; hence $F$ is identically zero. This proves the existence in Lemma 1.

The uniqueness of $g(x)$ is equivalent to the proposition:

*If* $g:R\to R$ *is continuous,* $g(0)=g(1)=0$,

*and* $g(x+y)=g(x)+g(y)$ *for all* $x,y$, *in* $R$, *then* $g(x)\equiv 0$.

This follows by a routine repetition of the technique above, since starting with $x=0$ and $x=1$, the operations $x\to x/2$ and $x,y\to x-y$ generate $S$, a subset dense in $R$, while $g(x/2)=g(x)/2$ and $g(x-y)=g(x)-g(y)$.

Finally if $g(0)=g(1)$ is not specified in Lemma 1, we may add $\alpha x$ to $g(x)$, for any real constant $\alpha$.

(Paper 77B1 & 2–375)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 1 & 2, January–June 1973

# Matroid Designs*

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**(November 22, 1972)**

Matroids are investigated in which equicardinality conditions are imposed on the flats. Such matroids are shown to be closely related to certain types of BIBD's. Existence and uniqueness theorems for these structures are derived, together with a boundedness criterion on their rank. Several classes are exhibited, including finite projective and affine geometries, certain $t$-designs (Steiner systems) and so-called trivioids. By viewing certain $t$-designs as matroids, new ways of constructing BIBD's are derived. Three new series of 3-designs and two new 4-designs are obtained by these methods. A matroid analysis of the 5-(24,8,1) design of Witt is presented, and examples are obtained from it of matroids having equicardinal hyperplanes but not equicardinal flats in lower ranks. Several general conjectures and existence problems for these types of matroids are suggested.

Key words: Balanced incomplete block designs; matroid designs; matroids.

## 1. Introduction

In this paper we shall investigate mathematical structures that bring together matroid theory and the theory of balanced incomplete block designs (BIBD's). From this unification we shall derive a number of new results in both of these fields. We shall be interested in two special types of matroids: those in which the hyperplanes are equicardinal, (called matroid designs), and those in which the flats of any given rank are equicardinal, (called perfect matroid designs). Matroid designs were first introduced under the name "equicardinal matroids" by U.S.R. Murty [10].[1]

Perfect matroid designs exhibit the high degree of regularity found in the classical finite projective and affine geometries; however, they comprise a variety of other structures, including $t$-designs (Steiner-systems) and so-called trivioids. They also represent a highly specialized class of BIBD's. Indeed, their complexity is indicated by the fact that the families of flats of any two different ranks form a BIBD. We shall develop a useful parametric description of perfect matroid designs that provides a classification of known examples and systematic framework for the search for new ones. Consistency conditions on the parameters are derived as well as additional types of existence conditions arising from the matroid structure. These conditions are used to tabulate all perfect matroid designs with less than forty-five points. We also explore the structure of perfect matroid designs, and show how certain known $t$-designs, when regarded as matroids, yield constructions of new $t$-designs.

Matroid designs in general may be viewed as variants of BIBD's. Frequently they consist of BIBD's that have been modified by the addition of certain objects and blocks. In this paper we show that matroid designs do indeed encompass a wider class of structures than perfect matroid designs, and we shall present a number of examples that will be derived from a matroid analysis of the famous 5-(24, 8, 1) design of Witt [13].

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*AMS subject classification:* Primary 05, Secondary 20.
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[1] Figures in brackets indicate the literature references at the end of this paper.

We begin with a development of the necessary theory, and then present the theory of perfect matroid designs. In the last section, we shall analyze the Witt design and give constructions of other types of matroid designs.

## 2. Matroids

A *matroid* $M=(E, \mathscr{I})$ is defined to be a finite set $E$ together with a nonempty set $\mathscr{I}$ of subsets of $E$, called *independent* sets, such that:
(I 1) $\varnothing$ is independent and every subset of an independent set is independent;
(I 2) for every $A \subseteq E$, all maximal independent subsets of $A$ have the same cardinality, called the *rank* of $A$, $r(A)$.

A maximal independent subset of $A$ is called a *basis of* $A$, or *$M$-basis of* $A$, if we wish to specify the matroid under consideration. *A basis of* $M$ is just an $M$-basis of $E$, and $r(M)$, the *rank of* $M$, is equal to $r(E)$. For convenience, it will always be assumed in the sequel that the function designated by $r$ is the rank function of the matroid designated by $M$.

A subset of $E$ that is not independent is said to be *dependent*, and the minimal dependent sets are called *circuits*.

A fundamental relation between independent sets and circuits is the following:
(1) THEOREM: *For any independent subset* J *of a matroid* $(E, \mathscr{I})$, *and any* $x \in E-J$, $J \cup \{x\}$ *contains at most one circuit.*
PROOF: Suppose to the contrary that $x$ is contained in two distinct circuits $C_1$ and $C_2$. Let $J'$ be any basis of $J \cup \{x\}$ containing $C_1 \cap C_2$. Then $|C_1-J'| \geq 1$ and $|C_2-J'| \geq 1$, so

$$|J'| \leq |J \cup \{x\}|-2 < |J|.$$

But then $J'$ and $J$ are two bases of $J \cup \{x\}$ of different cardinalities, contradicting (I 2).
(2) THEOREM: *Let* $\mathscr{C}$ *be a family of subsets of a finite set* E. $\mathscr{C}$ *is the circuit-family of a matroid on* E *if and only if*

(C1) $\varnothing \notin \mathscr{C}$ *and no member of* $\mathscr{C}$ *is a proper subset of another.*

(C2) *For any two distinct members* $C_1$ *and* $C_2$ *of* $\mathscr{C}$ *and* $x \in C_1 \cap C_2$, $C_1 \cup C_2-\{x\}$, *contains a member of* $\mathscr{C}$.

PROOF: Let $(E, \mathscr{I})$ be a matroid with circuit family $\mathscr{C}$. $\varnothing$ is independent, so $\varnothing \notin \mathscr{C}$. Then (C1) follows from the definition of circuit, and (C2) is an immediate consequence of (1). Conversely, let $\mathscr{C}$ satisfy (C1) and (C2), and let $\mathscr{I}$ be the family of subsets of $E$ containing no members of $\mathscr{C}$. Then $\varnothing \in \mathscr{I}$ and (I1) clearly holds. Suppose that (I2) does not hold for some subset $A$. Let $\mathscr{J}$ be the family of *maximum cardinality* members of $\mathscr{I}$ in $A$, *and* $\mathscr{J}'$ the family of all other *maximal* members of $\mathscr{I}$ in $A$. Let $|J_1 \cap J_2|$ be maximum among all $J_1 \in \mathscr{J}$, $J_2 \in \mathscr{J}'$. Then for $x \in J_2-J_1$, $J_1 \cup \{x\}$ contains a member $C$ of $\mathscr{C}$ containing $x$. If $C'$ were another such member of $\mathscr{C}$, then by (C2), $J_1$ would contain a member of $\mathscr{C}$, contradicting $J_1 \in \mathscr{I}$. Hence $C$ is unique. Further, $C \not\subseteq J_2$, so let $y \in C \cap (J_1-J_2)$. By the uniqueness of $C$, $J_1'=(J_1-\{y\}) \cup \{x\} \in \mathscr{I}$. Then $|J_1'|=|J_1|$, so $J_1' \in \mathscr{J}'$, but $|J' \cap J_2| > |J_1 \cap J_2|$, a contradiction. Hence $\mathscr{J}'=\varnothing$, and (I2) is proved.

Let $M=(E, \mathscr{I})$ be a matroid. For any $x \in E$ and $A \subseteq E$, $x$ is said to *depend* on $A$ if $x \in A$ or $\{x\} \cup A$ contains a circuit containing $x$. The *closure* of $A$, $cl(A)$, is the set of all elements that depend on $A$, and $A$ is said to be a *closed set*, or *flat*, if $cl(A)=A$. We write $cl_M(A)$ if we wish to emphasize that closure is taken with respect to $M$.
(3) It follows easily from the above definition that the intersection of a family of closed sets is closed. We may characterize the closure of a set as follows.
(4) THEOREM: *For any* $A \subseteq E$, cl (A) *is the unique maximal set* S *such that* $A \subseteq S \subseteq E$ *and* $r(A)=r(S)$.

PROOF: Let $S$ be some maximal set such that $A \subseteq S \subseteq E$ and $r(A) = r(S)$. Let $x \in cl(A) - A$, $C$ a circuit such that $x \in C \subseteq A \cup \{x\}$. By definition of circuit, $C - \{x\}$ is independent in $A$, hence it is contained in a basis $J$ of $A$. Since $r(A) = r(S)$ and $A \subseteq S$, $J$ is also a basis of $S$. However $J \cup \{x\}$ is dependent, hence $J$ is a basis of $S \cup \{x\}$ and $r(S \cup \{x\}) = r(S)$. By the maximality of $S$, it follows that $x \in S$. Hence $cl(A) \subseteq S$. On the other hand, let $x \in S - A$ and let $J$ be any basis of $A$. Then $J \cup \{x\}$ is dependent, so $J \cup \{x\}$ contains a circuit $C$ containing $x$; hence $x \in cl(A)$. Therefore $S \subseteq cl(A) \subseteq S$, so $S = cl(A)$, and $S$ is unique.

(5) COROLLARY: $cl(cl(A) \cup B) = cl(A \cup B)$ *for any subsets* A,B *of* E.

(6) A *k-flat* of a matroid $M = (E, \mathcal{I})$ is defined to be a closed subset of $E$ having rank $k$, or (by (4)) a maximal subset of $E$ having rank $k$. The *hyperplanes* of $M$ are its $(r(M) - 1)$ – flats. The following relationship between the hyperplanes and bases of $M$ is immediate.

(7) The hyperplanes of a matroid $M$ are the maximal subsets containing no basis, and the bases of $M$ are the minimal subsets contained in no hyperplane.

(8) THEOREM: *If* $\mathcal{B}$ *is the basis family of a matroid* $M = (E, \mathcal{I})$, *then* $\mathcal{B}^* = \{E - B: B \in \mathcal{B}\}$ *is the basis family of a matroid* $M^*$, *called the dual of* M. *The circuits of* $M^*$ *are precisely the set-complements of the hyperplanes of* M.

PROOF. Let $\mathcal{C}^* = \{E - H: H \text{ is a hyperplane of } M\}$. Clearly, $\mathcal{C}^*$ satisfies (C1).

Applying (7), we also see that the members of $\mathcal{B}^*$ are precisely the maximal subsets of $E$ containing no member of $\mathcal{C}^*$. For any distinct $C_1^*, C_2^* \in \mathcal{C}^*$ and $x \in C_1^* \cap C_2^*$, suppose $S = C_1^* \cup C_2^* - \{x\}$ contains no member of $\mathcal{C}^*$. Then $S \subseteq B^*$ for some $B^* \in \mathcal{B}^*$, and $x \notin B^*$. Letting $H_1 = E - C_1^*$, $H_2 = E - C_2^*$, and $B = E - B^*$, we have that $B - \{x\} \subseteq H_1 \cap H_2$. Since $r(B - \{x\}) = r(H_1) = r(H_2)$, it follows from (4) that $H_1 = H_2$ and $C_1^* = C_2^*$, a contradiction. Thus $\mathcal{C}^*$ also satisfies (C2), so it is the circuit family of a matroid $M^*$. By the first sentence above, it follows that $\mathcal{B}^*$ is the basis-family of $M^*$.

We note that $M^{**} = M$. The complements of the hyperplanes of $M$ (i.e., the circuits of $M^*$) are also called the *co-circuits* of $M$. Using (2) and (8), we may characterize the hyperplane-family of a matroid as follows.

(9) THEOREM: *Let* $\mathcal{H}$ *be a family of subsets of a finite set* E, $E \notin \mathcal{H}$. $\mathcal{H}$ *is the hyperplane family of a matroid on the set* E *if and only if*

(H1) *No member of* $\mathcal{H}$ *is a proper subset of another.*

(H2) *For any two distinct members* $H_1$ *and* $H_2$ *of* $\mathcal{H}$, *and any* $x \in E$, $\{x\} \cup (H_1 \cap H_2)$ *is contained in some member of* $\mathcal{H}$.

Although a matroid may be equally well characterized by its independent sets, its circuits, or its hyperplanes, we shall find it most useful to present it "geometrically" in terms of its hyperplanes, or more generally, its flats. We shall henceforth denote a matroid on the set $E$ with hyperplane-family $\mathcal{H}$ by $(E, \mathcal{H})$.

Let $M = (E, \mathcal{H})$ be any matroid.

(10) The 0-flat of $M$, $F^0$, is unique and consists of all elements not contained in any basis of $M$. Such elements are called *loops*. By the definition of closure, $F^0$ is a subset of every flat. Hence we may exclude all loops from $M$ without essentially altering the set relationships among the various flats.

(11) To simplify the presentation of later results, we shall therefore always assume in the sequel that $M$ has no loops.

(12) The 1-flats or *points*, of $M$ are analogous to geometric points in the sense that every flat $F$ is partitioned by the points that meet it. This follows immediately from (3), (4), and the assumption that the 0-flat of $M$ is null.

(13) A 2-flat of $M$ is called a *line*. For any two distinct points $a$ and $b$ of $M$, $r(a \cup b) = 2$, because $\{x, y\}$ is a basis of $a \cup b$ for any $x \in a$ and $y \in b$. Hence $a$ and $b$ are contained in a unique line of $M$, namely $cl(a \cup b)$.

The following theorem expresses a basic fact about the structure of flats in a matroid.

(14) THEOREM: *Let* M *be a rank* n *matroid. For any* i*-flat* $F^i$ *and* k*-flat* $F^k$ *of* M *such that* $F^i \subset F^k$,

$0 \leq i < k \leq n$, $F^k - F^i$ *is partitioned by the sets of form* $F^{i+1} - F^i$ *where* $F^{i+1}$ *is an* $(i+1)$*-flat containing* $F^i$ *and contained in* $F^k$.

PROOF: Let $F^i$ and $F^k$ be given as above, and let $J$ be a basis of $F^i$. For any $x \in F^k - F^i$ and basis $J$ of $F^i$, $J \cup \{x\}$ is independent, and, by (5), $cl(J \cup \{x\}) = cl(F^i \cup \{x\})$. Hence $F_x^{i+1} = cl(F^i \cup \{x\})$ is an $(i+1)$-flat containing $F^i$ and $x$. If $F_x^{i+1} \not\subseteq F^k$, then $F_x^{i+1} \cap F^k$ would be a flat properly containing the $i$-flat $F^i$, and properly contained in the $(i+1)$-flat $F_x^{i+1}$, which is absurd. Hence every element $x \in F^k - F^i$ is contained in an $(i+1)$-flat $F_x^{i+1}$ such that $F^i \subseteq F_x^{i+1} \subseteq F^k$. Finally, if $A$ and $B$ are any two distinct $(i+1)$-flats that contain $F^i$ and are contained in $F^k$, then $A \cap B$ is a flat containing the $i$-flat $F^i$ and properly contained in the $(i+1)$-flat $A$. Hence $A \cap B = F^i$, and $(A - F^i) \cap (B - F^i) = \emptyset$. Thus the sets of form $F^{i+1} - F^i$, where $F^{i+1}$ is an $(i+1)$-flat containing $F^i$ and contained in $F^k$, partition $F^k - F^i$.

## 3. Perfect Matroid Designs

### 3.1. The Geometry of Flats in a Perfect Matroid Design

The geometric structure imposed on the flats of a matroid by (14) is not nearly as restrictive as the structure of the classical finite affine and projective geometries. We propose to investigate matroids that bear a closer resemblance to the classical geometries, in that equicardinality conditions are imposed on their flats. We shall find that these structures constitute a very special and intriguing class of block designs.

(15) A matroid $M$ is said to be a *matroid design* if all of its hyperplanes have the same cardinality, which we denote by $k(M)$. If, for every integer $j$, $0 \leq j \leq r(M)$, all of the $j$-flats of $M$ have the same cardinality $\alpha(j)$, then $M$ is called a *perfect matroid design*, abbreviated by *PMD*. Like the finite geometries whose properties it generalizes, PMD's exhibit strong regularities of structure in addition to the equicardinality of the flats of a given rank.

(16) THEOREM: *Let* M *be a rank* n *PMD. For any integers* i, j, k, *such that* $0 \leq i \leq j \leq k \leq n$, *and any* i*-flat* $F^i$ *and* k*-flat* $F^k$ *such that* $F^i \subseteq F^k$, *the number* $t_M(i, j, k)$ *of* j*-flats* $F^j$ *such that* $F^i \subseteq F^j \subseteq F^k$ *is independent of the choice of* $F^i$ *and* $F^k$. *Moreover, the function* $t_M(i, j, k)$ *so defined satisfies the following relations.*

(T0) $$t_M(i, i, k) = 1, \quad 0 \leq i \leq k \leq n,$$

*and*

$$t_M(0, 1, i+1) > t_M(0, 1, i), \quad 1 \leq i \leq n-1.$$

(T1) $$t_M(i, i+1, k) = \frac{t_M(0, 1, k) - t_M(0, 1, i)}{t_M(0, 1, i+1) - t_M(0, 1, i)}, \quad 0 \leq i < k \leq n.$$

(T2) $$t_M(i, j, k) = \frac{t_M(i, l, k) t_M(l, j, k)}{t_M(i, l, j)}, \quad 0 \leq i \leq l \leq j \leq k \leq n.$$

PROOF: For given $i$ and $k$, $0 \leq i \leq k \leq n$, and any given $i$-flat $F^i$ and $k$-flat $F^k$ such that $F^i \subseteq F^k$, $F^i$ is clearly the only $i$-flat containing $F^i$ and contained in $F^k$. Hence $t_M(i, i, k) = 1$ for every choice of $F^i$ and $F^k$. Moreover, where $F^{i+1}$ is any $(i+1)$-flat containing $F^i$, we have $F^i \subset F^{i+1}$, and hence, by (12), $F^{i+1}$ contains more points than does $F^i$.

(17) Where $\alpha(k)$ is the number of elements in any $k$-flat of $M$, the number of points in any $k$-flat of $M$ is clearly $\alpha(k)/\alpha(1)$, and hence $t_M(0, 1, k)$ is well-defined for every $k$, $1 \leq k \leq n$. For any $i$ and

$k$, $1 \leq i < k \leq n$, and for any $i$-flat $F^i$ and $k$-flat $F^k$ such that $F^i \subseteq F^k$, it follows from (14) that the number of $(i+1)$-flats $F^{i+1}$ such that $F^i \subseteq F^{i+1} \subseteq F^k$ is

$$\frac{\alpha(k)-\alpha(i)}{\alpha(i+1)-\alpha(i)} = \frac{t_M(0,1,k)-t_M(0,1,i)}{t_M(0,1,i+1)-t_M(0,1,i)},$$

which is independent of the particular $F^i$ and $F^k$ chosen. Thus we have shown that $t_M(i, i+1, k)$ is well-defined for every $i$ and $k$, $0 \leq i < k \leq n$, and that (T1) holds.

Now let $i$ be fixed, $0 \leq i \leq n$. We shall prove by induction on $j$, $i \leq j \leq n$, that for every $k$, $j \leq k \leq n$, $t_M(i, j, k)$ is well-defined. We have already shown this for $j = i$ and $j = i+1$. Suppose we have shown that $t_M(i, j, k)$ is well-defined for every $j$, $i \leq j \leq j_0$. If $j_0 = n$, we are done. Otherwise, choose $k \geq j_0+1$ and let $F^i$ be an $i$-flat, $F^k$ a $k$-flat such that $F^i \subseteq F^k$. Let $V^{j_0}$ denote the set of $j_0$-flats $F^{j_0}$ such that $F^i \subseteq F^{j_0} \subseteq F^k$, and let $V^{j_0+1}$ denote the set of $(j_0+1)-$ flats $F^{j_0+1}$ such that $F^i \subseteq F^{j_0+1} \subseteq F^k$.

(18) Every member of $V^{j_0}$ is contained in $t_M(j_0, j_0+1, k)$ members of $V^{j_0+1}$, by (17), and every member of $V^{j_0+1}$ contains $t_M(i, j_0, j_0+1)$ members of $V^{j_0}$, by the inductive hypothesis on $j$. Since $|V^{j_0}| = t_M(i, j_0, k)$, also by the hypothesis on $j$, we have

$$|V^{j_0+1}| = \frac{t_M(i,j_0,k)\,t_M(j_0,j_0+1,k)}{t_M(i,j_0,j_0+1)},$$

and hence $t_M(i, j_0+1, k) = |V^{j_0+1}|$ is well-defined. This proves by induction that $t_M(i, j, k)$ is well-defined whenever $0 \leq i \leq j \leq k \leq n$.

Finally, (T2) follows from a counting argument similar to that used in (18).

(19) The function $t_M(i, j, k)$ of (16) is called the *t-function of M*.

### 3.2. *t*-Designs

Perfect matroid designs form a special class of structures called $t$-designs, which are a generalization of balance incomplete block designs (BIBD's). (They have also been called tactical configurations by Hanani [6]. See also [4] and [7].)

(20) For $t \geq 2$ let $t$ be an integer greater than 1. A *t-design*, or more specifically, a $t-(v, k, \lambda)$ *design*, $(V, \mathscr{W})$, is a $v$-set $V$ and a system $\mathscr{W}$ of $k$-subsets of $V$, $k < v$, called *blocks*, such that every $t$-subset of $V$ is contained in exactly $\lambda$ blocks.

Repeated blocks are admissible in the system $\mathscr{W}$. We shall sometimes refer to any such pair $(V, \mathscr{W})$ as a $D_t(v, k, \lambda)$. A BIBD is precisely a 2-design.

(21) For any $t-(v, k, \lambda)$ design $(V, \mathscr{W})$ and integer $i$, $0 \leq i \leq t$, we let $\lambda_i$ denote the number of blocks containing any fixed $i$-subset of $V$. $\lambda_t$ is identical with $\lambda$, and, in general (see, for example, [7]),

$$\lambda_i = \lambda \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}} \quad \text{for } 0 \leq i \leq t. \tag{22}$$

In particular, $(V, \mathscr{W})$ is also an $i-(v, k, \lambda_i)$ design for $2 \leq i \leq t$. Moreover, it follows that, for a given set of parameters $t$, $v$, $k$, $\lambda$, the numbers $\lambda_i$ given by (22) must be integers for a $t-(v, k, \lambda)$ design to exist.

(23) A particularly useful result is the well-known *Fisher Inequality*, which states that, if a $t-(v, k, \lambda)$ design exists, then the number of blocks, $\lambda_0$, is greater than or equal to the number of elements, $v$. Using (22), we may then state this condition as

$$\frac{\lambda\binom{v}{t}}{\binom{k}{t}} \geq v. \tag{24}$$

### 3.3. Consequences of Fisher's Inequality for PMD's

Let $M$ be a rank $n$ PMD. For any $i$-flat $F^i$ and $k$-flat $F^k$ of $M$ such that $F^i \subseteq F^k$, let $I_j(F^i, F^k)$ denote the set of all $j$-flats $F^j$ of $M$ such that $F^i \subseteq F^j \subseteq F^k$.

(25) THEOREM: *For any PMD* M *and integers*, i, j, k, *such that* $2 \leq i+2 \leq j < k \leq r(M)$, *and any* i-*flat* $F^i$ *and* k-*flat* $F^k$ *of* M *such that* $F^i \subseteq F^k$, *the set* $\mathscr{W} = \{I_{i+1}(F^i, F^j)$: $F^i \subseteq F^j \subseteq F^k$ *and* $F^j$ *is a* j-*flat of* M$\}$ *is the system of blocks of a BIBD on the set* $V = I_{i+1}(F^i, F^k)$ *with* $\lambda = t_M(i+2, j, k)$.

PROOF: For any member $I_{i+1}(F^i, F^j)$ of $\mathscr{W}$, $F^i \subseteq F^j \subsetneq F^k$ because $j < k$. Hence, for $x \in F^k - F^j$, the $(i+1)$-flat $A = cl(F^i \cup \{x\})$ is contained in $V$, but is not contained in $I_{i+1}(F^i, F^j)$. Thus the members of $\mathscr{W}$ are proper subsets of $V$.

For any two distinct $(i+1)$-flats $F_1$ and $F_2$ in $V$, $F_1 \cap F_2 = F^i$, by (14). Let $J$ be a basis of $F^i$, and let $x \in F_1 - F^i$, $y \in F_2 - F^i$. Then $J \cup \{x\}$ is a basis of $F_1$, and $J \cup \{y\}$ is a basis of $F_2$. Since $x \notin F_2 = cl(J \cup \{y\})$, $J \cup \{x, y\}$ is an independent set of rank $i+2$ in $F_1 \cup F_2$, and in fact it must be a basis of $F_1 \cup F_2$, because every element of $F_1 \cup F_2$ depends on either $J \cup \{x\}$ or on $J \cup \{y\}$. Hence $cl(F_1 \cup F_2)$ is the unique $(i+2)$-flat containing $F_1 \cup F_2$, and $cl(F_1 \cup F_2) \subseteq F^k$. Further, every $(i+2)$-flat is contained in exactly $t_M(i+2, j, k)$ $j$-flats contained in $F^k$. Hence every two $(i+1)$-flats in $V$ are contained together in exactly $t_M(i+2, j, k)$ $j$-flats contained in $F^k$, which proves that $(V, \mathscr{W})$ is a BIBD with $\lambda = t_M(i+2, j, k)$.

(26) In (25), the members of $\mathscr{W}$ are in one-to-one correspondence with the $j$-flats $F^j$ such that $F^i \subseteq F^j \subseteq F^k$. Hence $|\mathscr{W}| = t_M(i, j, k)$. Likewise, $|V| = t_M(i, i+1, k)$. Applying Fisher's Inequality to the results of (25) and (26), we obtain, for any perfect matroid design $M$

(T3) $t_M(i, i+1, k) \leq t_M(i, j, k)$ for $2 \leq i+2 \leq j < k \leq n$.

(27) A function $t(i, j, k)$ of integers $i, j, k$ such that $0 \leq i \leq j \leq k \leq n$ for some integer $n$ is said to be *T-consistent* if it is positive integer-valued and satisfies the relations (T0), (T1), (T2) of (15) and (T3) above. In particular, the $t$-function of a PMD is $T$-consistent.

(28) Any T-consistent function that is actually the $t$-function of some PMD will be called a *realizable* $T$-consistent function.

### 3.4. Further Properties of Matroids and Perfect Matroid Designs: Reduction, Contraction, Truncation

Let $M = (E, \mathscr{I})$ be a fixed matroid of rank $n$.

(29) For any subset $E'$ of $E$, let $\mathscr{I}' = \{J \in \mathscr{I}: J \subseteq E'\}$. $M' = (E', \mathscr{I}')$ clearly satisfies (I1) and (I2), hence it is a matroid, called the *submatroid of M on* $E'$, or the *reduction of M to* $E'$, and denoted by $M \times E'$. Clearly, the rank function, $r'$, of $M'$ is just the rank function of $M$ restricted to the subsets of $E'$. The following result is immediate.

(30) THEOREM: *The flats of* $M \times E'$ *are precisely the intersections of the flats of* M *with* $E'$.

(31) Let $E'$ be a subset of $E$, and let $\mathscr{I}' = \{J' \subseteq E': J' \cup J \in \mathscr{I}$ for some $M$-basis $J$ of $E - E'\}$. It is easily verified that $(E', \mathscr{I}')$ satisfies (I1) and (I2), hence it is a matroid, called the *contraction of M to* $E'$, and denoted by $M \cdot E'$.

(32) For any $A \subseteq E'$, the rank of $A$ in $M \cdot E'$ is given by $r'(A) = r(A \cup (E - E')) - r(E - E')$, where $r$ is the rank function of $M$. In particular, $r(M \cdot E') = r(M) - r(E - E')$.

The flats of $M \cdot E'$ are characterized in terms of the flats of $M$ as follows.

(33) THEOREM: *For* $E' \subseteq E$ *and* $0 \leq j \leq r(M) - r(E - E')$, $F \subseteq E'$ *is a* j*-flat of* $M \cdot E'$ *if and only if* $F \cup (E - E')$ *is a* $(j + r(E - E'))$*-flat of* M.

PROOF: A set $F \subseteq E'$ is closed in $M \cdot E'$ if and only if $r'(F \cup \{x\}) > r'(F)$ for every $x \in E' - F$, by (4). This occurs if and only if $r(F \cup (E - E') \cup \{x\}) > r(F \cup (E - E'))$ for every $x \in E - (F \cup (E - E'))$ that is, if and only if $F \cup (E - E')$ is closed in $M$. Further, by (32), $r'(F) = j$ if and only if $r(F \cup (E - E')) = j + r(E - E')$. Thus $F$ is a $j$-flat of $M \cdot E'$ if and only if $F \cup (E - E')$ is a $(j + r(E - E'))$-flat of $M$.

(34) Let $F^i \subseteq F^k$ be an $i$-flat and $k$-flat respectively. *The interval of* $M$ *between* $F^i$ *and* $F^k$, $M(F^i, F^k)$, is the matroid $(M \times F^k) \cdot (F^k - F^i)$.

Since the set $\mathfrak{F}$ of flats of a matroid $M$ includes $E$ and is closed under intersections, $\mathfrak{F}$ may be viewed as a lattice, ordered by inclusion. Such a lattice is called a *matroid lattice*, or *geometric lattice* [1].

(35) By (30) and (33), the flats of an interval $M(F^i, F^k)$ of $M$ correspond $1-1$ with the flats $A$ of $M$ such that $F^i \subseteq A \subseteq F^k$, and the corresponding lattices ordered by inclusion are isomorphic. This fact is sometimes useful in picturing the structure of a matroid interval $M(F^i, F^k)$. In particular, this observation yields a proof of the following.

(36) THEOREM: *If* M *is a rank* n *PMD with* t*-function* $t_M(i, j, k)$, *and* $F^h \subseteq F^l$ *are an* h*-flat and* l*-flat of* M *respectively, then* $M(F^h, F^l)$ *is a rank* $l$-h *PMD with* t*-function*

$$t(i, j, k) = t_M(i + h, j + h, k + h), 0 \leq i \leq j \leq k \leq l\text{-h}.$$

(37) For any integer $l$, $0 \leq l \leq r(M)$, let $\mathscr{I}' = \{J \subseteq E: J \in \mathscr{I} \text{ and } |J| \leq l\}$. Clearly, $M' = (E, \mathscr{I}')$ is a matroid having rank function $r'(A) = \min\{l, r(A)\}$. $M'$ is called the *l-truncation of* $M$, and denoted by $M^{(l)}$. The flats of $M^{(l)}$ are the flats of $M$ having rank at most $l-1$, and $E$. If $M$ is a PMD having $t$-function $t_M(i, j, k)$, then $M^{(l)}$ is a rank $l$ PMD having $t$-function

$$t(i, j, k) = t_M(i, j, k) \text{ for } 0 \leq i \leq j \leq k < l,$$

$$t(i, j, l) = t_M(i, j, n) \text{ for } 0 \leq i \leq j < l,$$

and

$$t(i, l, l) = 1 \text{ for } 0 \leq i \leq l.$$

(38) A point (1-flat) of a matroid $M$ is said to be an *m-point* if it has cardinality $m$, and it is a *simple point* if $m = 1$. In general, a *simple flat* of $M$ is a flat that is an independent set, i.e., a flat whose rank equals its cardinality. The matroid $M$ is said to be *simple* if all of its points are simple.

(39) Let $M = (E, \mathscr{H})$ be any matroid, and $\alpha$ a positive integer. For each element $x \in E$ choose a set $S_x$ of $\alpha$ elements in such a way that for distinct elements $x, x' \in E$, $S_x \cap S_{x'} = \emptyset$. Let $E' = \bigcup_{x \in E} S_x$, and let $\mathscr{H}' = \{\bigcup_{x \in H} S_x : H \in \mathscr{H}\}$. Clearly, $M' = (E', \mathscr{H}')$ satisfies (H1) and (H2), hence $M'$ is a matroid. $M'$ is said to be an *α-inflation* of $M$, and $M$ is an *α-deflation* of $M'$. Now $F'$ is a flat of $M'$ if and only if $F' = \bigcup_{x \in F} S_x$ for some flat $F$ of $M$. This relation establishes a one-to-one correspondence between the flats of $M$ and $M'$ that preserves inclusion. It follows that if $M$ is a PMD whose points have cardinality $\alpha$, then there exists an $\alpha$-deflation $M'$ of $M$ such that $M'$ is simple, and the lattices of $M$ and $M'$ are isomorphic. In particular, $M$ is a PMD if and only if $M'$ is, and their $t$-functions are the same. Hence there is no loss of generality in proving certain results for simple PMD's.

### 3.5. *d*-Sequences and *D*-Consistency

The relations (T0)–(T3) impose strong restrictions on the possible parameters for a PMD. However, it is often quite awkward to verify directly whether a given function $t(i, j, k)$ satisfies

these relations. We shall define an auxiliary set of parameters for a PMD that is considerably easier to handle, and contains all the information of the associated $t$-function.

Let $M$ be a rank $n$ PMD with $t$-function $t_M(i, j, k)$, $0 \leq i \leq j \leq k \leq n$, and define the sequence $d(M) = (d_n, d_{n-1}, \ldots, d_1)$ by the relation

$$d_i = t_M(0, 1, i) - t_M(0, 1, i-1), \qquad 1 \leq i \leq n, \tag{40}$$

where we take $t_M(0, 1, 0) = 0$. In case $r(M) = 0$, that is, if $M = \{\emptyset, \emptyset\}$, we let $d(M)$ be the empty sequence. $d(M)$ is called the *d-sequence* of $M$.

(41) THEOREM: *Let* t(i, j, k) *be a positive, integer-valued,* T*-consistent function on integers* i, j, k *where* $0 \leq i \leq j \leq k \leq n$. *Let* $d_i = t(0, 1, i) - t(0, 1, i-1)$ *for each* $i$, $1 \leq i \leq n$. *Then*

$$t(i, j, k) = \frac{\displaystyle\prod_{l=i+1}^{j} \sum_{m=l}^{k} d_m}{\displaystyle\prod_{l=i+1}^{j} \sum_{m=l}^{k} d_m}, \text{ for } 0 \leq i \leq j \leq k \leq n. \tag{42}$$

*where an empty product is assumed to take the value* 1 .

PROOF: For given $i$, $0 \leq i \leq n$, the proof of (41) is by induction on the set $S = \{(k, j): i \leq j \leq k \leq n\}$, ordered lexicographically. For $k = j = i$, both sides of (42) are equal to "1".

Assume then that (42) holds for all pairs $(k, j) \in S$ such that $(k, j) \leq (w, u)$ for some $(w, u) \in S$. We may assume that $(w, u) \neq (n, n)$; otherwise, we are done. Suppose first that $w = u$. The successor of $(u, u)$ in $S$ is $(u+1, i)$, hence we must show that (42) holds for $j = i$, $k = u+1$. This follows immediately from (T0) and the fact that empty products have value "1". Hence we may assume that $u < w$, and we must prove (42) for $(k, j) = (w, u+1)$. By (T2) and (T1),

$$t(i, u+1, w) = \frac{t(i, u, w)t(u, u+1, w)}{t(i, u, u+1)} = \frac{t(i, u, w)}{t(i, u, u+1)} \cdot \frac{t(0, 1, w) - t(0, 1, u)}{t(0, 1, u+1) - t(0, 1, u)},$$

$$= \frac{t(i, u, w)}{t(i, u, u+1)} \cdot \frac{\displaystyle\sum_{m=u+1}^{w} d_m}{d_{u+1}},$$

by definition of the $d_i$'s.

Applying the induction hypothesis to $t(i, u, w)$ and $t(i, u, u+1)$ in the above expression and simplifying, we have finally

$$t(i, u+1, w) = \frac{\displaystyle\sum_{m=u+1}^{w} d_m \prod_{l=i+1}^{u} \sum_{m=l}^{w} d_m}{d_{u+1} \displaystyle\prod_{l=i+1}^{u} \sum_{m=l}^{u+1} d_m} = \frac{\displaystyle\prod_{l=i+1}^{u+1} \sum_{m=l}^{w} d_m}{\displaystyle\prod_{l=i+1}^{u+1} \sum_{m=l}^{u+1} d_m},$$

which is (42) with $(k, j) = (w, u+1)$. The theorem follows now by induction.

A sequence $(d_n, d_{n-1}, \ldots, d_1)$ of positive integers is said to be *D-consistent* if $d_1 = 1$ and the associated function $t(i, j, k)$ given by (42) is $T$-consistent. The relations (T0)–(T3) for the function $t(i, j, k)$ then translate into the following conditions on $(d_n, d_{n-1}, \ldots, d_1)$.

(43) THEOREM: *A sequence* $D = (d_n, d_{n-1}, \ldots, d_1)$ *is* D*-consistent if and only if*

$$d_1 = 1 \text{ and } d_i \geq 1, \ 1 \leq i \leq n. \tag{D0}$$

$$\frac{\displaystyle\prod_{l=i+1}^{j} \sum_{m=l}^{k} d_m}{\displaystyle\prod_{l=i+1}^{j} \sum_{m=l}^{j} d_m} \text{ is integral for } 0 \leq i \leq j \leq k \leq n. \tag{D1}$$

(D2) $$d_{i-1} | d_i,\ 2 \leq i \leq n,\ \textit{and}\ \frac{d_{i+1}}{d_i} \geq \frac{d_i}{d_{i-1}},\ 2 \leq i \leq n-1.$$

PROOF: Let $t(i, j, k)$ be given from $D$ by (42). In particular,

(44) $$t(0, 1, i) = \sum_{m=1}^{i} d_m/d_1 = \sum_{m=1}^{i} d_m,$$

hence

(45) $$d_i = t(0, 1, i) - t(0, 1, i-1) \text{ for } 1 \leq i \leq n,$$

where we take $t(0, 1, 0) = 0$.

It follows from (45) that $d_i \geq 1$ if and only if $t(0, 1, i) > t(0, 1, i-1)$. Further, $t(i, i, k) = 1$, by our convention that empty products take the value '1'. Hence (D0) is equivalent to (T0). (D1) is equivalent to the requirement that $t(i, j, k)$ be integer valued.

To prove (D2), given that $t(i, j, k)$ is $T$-consistent, we use (45) to write

(46) $$\frac{d_{i+1}}{d_i} = \frac{[t(0, 1, i+1) - t(0, 1, i-1)] - [t(0, 1, i) - t(0, 1, i-1)]}{t(0, 1, i) - t(0, 1, i-1)}$$

$$= t(i-1, i, i+1) - 1 \text{ , for } 1 \leq i \leq n-1, \text{ by (T1).}$$

Likewise,

(47) $$\frac{d_i}{d_{i-1}} = t(i-2, i-1, i) - 1 \text{ for } 2 \leq i \leq n.$$

In particular,

(48) $$d_{i-1} | d_i \text{ for } 2 \leq i \leq n.$$

By (T2), $t(i-2, i, i+1) = \dfrac{t(i-2, i-1, i+1)t(i-1, i, i+1)}{t(i-2, i-1, i)}$ and by (T3) $t(i-2, i, i+1)$ $\geq t(i-2, i-1, i+1)$. Hence, using the fact that $t(i, j, k)$ is positive,

(49) $$t(i-1, i, i+1) \geq t(i-2, i-1, i).$$

Substituting (46) and (47) in (49), we find that $\dfrac{d_{i+1}}{d_i} \geq \dfrac{d_i}{d_{i-1}}$, $2 \leq i \leq n-1$. Combined with (48), this establishes (D2).

It remains to show that (T1), (T2), and (T3) hold for $t(i, j, k)$ given that D satisfies (D0)–(D2). From (42) and (44) it follows that

$$\frac{t(0, 1, k) - t(0, 1, i)}{t(0, 1, i+1) - t(0, 1, i)} = \frac{\sum_{m=i+1}^{k} d_m}{d_{i+1}} = t(i, i+1, k),$$

which establishes (T1). (T2) also follows immediately from (42) by substitution.

To prove (T3), we write, for integers $2 \leq i+2 \leq j < k \leq n$,

(50) $$t(i,j,k)=\frac{\prod_{l=i+1}^{j}\sum_{m=l}^{k}d_m}{\prod_{l=i+1}^{j}\sum_{m=l}^{j}d_m}=\frac{\sum_{m=i+1}^{k}d_m}{d_j}\cdot\prod_{l=i+2}^{j}\left(\frac{\sum_{m=l}^{k}d_m}{\sum_{m=l-1}^{j}d_m}\right)$$

By (D2), $d_m \geq d_{m-1}\frac{d_l}{d_{l-1}}$ for $m \geq l$. Hence (50) implies

(51) $$t(i,j,k)\geq\frac{\sum_{m=i+1}^{k}d_m}{d_j}\left(\prod_{l=i+2}^{j}\frac{d_l}{d_{l-1}}\right)\left(\prod_{l=i+2}^{j}\frac{\sum_{m=l}^{k}d_{m-1}}{\sum_{m=l-1}^{j}d_m}\right)$$

The first iterated product is equal to $\frac{d_j}{d_{i+1}}$, and the second is greater than or equal to "1" (using the positivity of the $d_m$'s and the fact that $j<k$). Hence

(52) $$t(i,j,k)\geq\frac{\sum_{m=i+1}^{k}d_m}{d_{i+1}}=t(i,i+1,k).$$

COROLLARY: *For any PMD* M, *the* d-*sequence of* M, d(M), *satisfies* (*D0*), (*D1*), (*D2*), *and* $t_M$(i, j, k) *is given in terms of* d(M) *by* (42).

### 3.6. Classes of Perfect Matroid Designs and Their *d*-Sequences

We shall first describe the operations on $d$-sequences that correspond to the operations introduced in Section 3.4.

Let $M$ be a rank $n$ PMD with $d$-sequence $d(M)=(d_n, d_{n-1}, \ldots, d_1)$.

(53) From paragraph (39) it follows that an $\alpha$-inflation or $\alpha$-deflation of $M$ has the same $d$-sequence as $M$.

(54) If $F^p$ is a $p$-flat and $F^q$ a $q$-flat of $M$ such that $F^p \subseteq F^q$, then the interval of $M$, $M(F^p, F^q)$ has $t$-function $t(i,j,k)=t_M(i+p, j+p, k+p)$, $0\leq i\leq j\leq k\leq q-p$, by (36). Hence, for $1\leq k\leq q-p$, $1\leq k\leq q-p$,

$$t(0,1,k)-t(0,1,k-1)=t_M(p,p+1,k+p)-t_M(p,p+1,k+p-1)$$

$$=\frac{\sum_{m=p+1}^{k+p}d_m}{d_{p+1}}-\frac{\sum_{m=p+1}^{k+p-1}d_m}{d_{p+1}}=\frac{d_{k+p}}{d_{p+1}},\text{ by (42).}$$

Hence, $M(F^p, F^q)$ has $d$-sequence $\left(\frac{d_q}{d_{p+1}}, \frac{d_{q-1}}{d_{p+1}}, \ldots, \frac{d_{p+1}}{d_{p+1}}\right)$.

(55) For any integer $h$, $1\leq h\leq n$, the $d$-sequence of $M^{(h)}$ (the $h$-truncation of $M$) is clearly

$$\left(\sum_{j=h}^{n}d_j, d_{h-1}, \ldots, d_1\right).$$

(56) A matroid $M$ on the set $E$ is said to be *disconnected* if for some nonempty proper subset $S$ of $E$, every circuit of $M$ is contained in $S$ or in $E-S$. Such a subset is called a *proper separator* of $M$. If $M$ contains no circuits, it is said to be *totally disconnected.*

(57) THEOREM: *If a simple PMD is disconnected, then it is totally disconnected.*

PROOF: Let $M$ be a disconnected, simple PMD and let $S$ be a proper separator of $M$. Let $C$ be a circuit of $M$ having minimum cardinality, say $|C| = \gamma$. $\gamma \geq 3$ because $M$ is simple and contains no loops. We may also suppose without loss of generality that $C \subseteq S$. $r(C) = \gamma - 1$, so $cl(C)$ is a $(\gamma-1)$-flat and $|cl(C)| \geq \gamma \geq 3$.

Let $x, y \in C$, $x \neq y$. Then $C-\{x, y\}$ is independent, and $I = (C-\{x, y\}) \cup \{z\}$ is independent for any $z \in E-S$, since no circuit of $M$ meets both $S$ and $E-S$ and $M$ contains no loops. Let $w$ be an element of $E-I$ that depends on $I$. Then $w$ depends either on $\{z\}$ or on $C-\{x, y\}$. The former is impossible because $M$ is simple. If the latter were the case, then $(C-\{x, y\}) \cup \{w\}$ would contain a circuit smaller than $C$, which is contrary to the choice of $C$. Thus no element of $E-I$ depends on I, so $cl\ (I) = I$. $I$ is therefore a $(\gamma-1)$-flat having cardinality $\gamma-1$. But then $I$ and $cl(C)$ are two $(\gamma-1)$-flats having distinct cardinalities, contrary to the definition of a PMD. Hence $M$ contains no circuits; that is, $M$ is totally disconnected.

We shall now introduce three general classes of PMD's.

(58) For any integers $v$ and $k$ such that $v > k \geq 0$, there exists a PMD $M$ on $v$ elements with hyperplane size $k$. Namely, let $M_E$ denote the totally disconnected matroid on a $v$-set $E$, and let $M = M_E^{(k+1)}$ (the $(k+1)$-truncation of $M_E$). Where $\alpha$ is a positive integer, any $\alpha$-inflation $M(\alpha)$ of such an $M$ is called an $(\alpha, k, v)$-*trivioid*, denoted by $\sigma(\alpha, k, v)$, and $M(\alpha)$ is said to be a *trivial* matroid design. The hyperplanes of $M(\alpha)$ are precisely the subsets of $E$ that are unions of any $k$ distinct points.

The $d$-sequences of trivioids are characterized as follows.

(59) Theorem: *Every sequence of form* $(d_n, 1, 1, \ldots, 1)$ *having length* n *is the* d*-sequence of an* $(\alpha, n-1, d_n+n-1)$*-trivioid, and conversely.*

PROOF: Let $d(M) = (d_n, 1, \ldots, 1)$ for some rank $n$ PMD $M = (E, \mathscr{H})$. By (42), $t_M(0, 1, j) = j$ for $0 \leq j \leq n-1$, and $t_M(0, 1, n) = d_n + n - 1$.

Let $\alpha$ be the common cardinality of the points of $M$, and let $M'$ be an $\alpha$-deflation of $M$. $M'$ is simple and has the same $t$-function as $M$. Thus, every $j$-flat of $M'$ has cardinality $j$, for $0 \leq j \leq n-1$. Since every $j$-subset of $E$ is certainly contained in some $j$-flat, it follows that every $j$-subset of $E$ is a $j$-flat for $0 \leq j \leq n-1$. Thus $M'$ is a $(1, n-1, d_n+n-1)$-trivioid, and $M$ is an $(\alpha, n-1, d_n+n-1)$-trivioid.

Conversely, for any positive integers $n$, $d_n$, and $\alpha$, an $(\alpha, n-1, d_n+n-1)$-trivioid certainly exists, and it is straightforward to verify that its $d$-sequence is $(d_n, 1, \ldots, 1)$.

(60) Any sequence of form $(d_n, 1, 1, \ldots, 1)$ is said to be a *trivial* sequence. By (59), every trivial sequence is $D$-consistent.

(61) Let us also note here a general property of $D$-consistent sequences: if $D = (d_n, d_{n-1}, \ldots, d_1)$ is $D$-consistent and $d_j = 1$ for some $j$, then $d_j = d_{j-1} = \ldots = d_1 = 1$. Indeed, for $1 \leq i \leq j$, (D0) implies that $d_i \geq 1$, and (D2) implies that $d_i \mid d_j$. Hence $d_j = 1$ implies that $d_i = 1$ for $1 \leq i \leq j$.

(62) We recall that for any matroid $M = (E, \mathscr{H})$, the co-circuits of $M$ are the complements in $E$ of the hyperplanes of $M$. Since any hyperplane of $M$ is partitioned by the points that meet it, the same holds for any co-circuit of $M$. Let $c^*(M)$ denote the minimum cardinality of the co-circuits of $M$. If $M$ is a matroid design, then $c^*(M)$ is the size of every co-circuit of $M$. In particular, if $M$ is a *simple* PMD, then the leading term of $d(M)$, the $d$-sequence of $M$, is $c^*(M)$.

(63) THEOREM: *Every PMD* M *with prime co-circuit cardinality* $c^*(M)$ *is a trivioid.*

PROOF: Let $c^*(M) = p$ be a prime. Where $\alpha$ is the cardinality of the points of $M$, $\alpha \mid c^*(M)$ because every co-circuit is partitioned by the points that meet it. If $\alpha = p$, then the complement of any point is a hyperplane, so $M$ is a $\left(p, \frac{k(M)}{p}, \frac{k(M)}{p}+1\right)$-trivioid. Otherwise, $\alpha = 1$ *and* $M$ is simple. Therefore $d(M) = (p, d_{n-1}, \ldots, d_1)$. Let $j$ be the greatest integer such that $d_j = 1$. By (D2), $d_{n-1} \mid p$ and $\frac{p}{d_{n-1}} \geq \frac{d_{j+1}}{d_j} > 1$. Hence $d_{n-1} = 1$, so $d_i = 1$ for $1 \leq i \leq n-1$. Thus, $d(M)$ is trivial, so, by (59), $M$ is a trivioid.

In another paper [14] we show that Theorem (63) applies to matroid designs in general.

A second class of PMD's comes from the $t$-designs with $\lambda = 1$.

(64) THEOREM: *For any integer* $t \geq 2$, M *is a* $t-(v, k, 1)$ *design such that* $k < v$ *if and only if* M *is a simple rank* $t+1$ *PMD with* d*-sequence* $(v-k, k-t+1, 1, \ldots, 1)$.

PROOF: Let $t \geq 2$, and let $M=(E, \mathscr{H})$ be a $t-(v, k, 1)$ design such that $k < v$. First we shall show that $(E, \mathscr{H})$ satisfies (H1) and (H2). Since $k < v$, the members of $\mathscr{H}$ are proper subsets of $E$. Further, no member of $\mathscr{H}$ is properly contained in another. Thus, (H1) is satisfied.

Suppose that $H_1$ and $H_2$ are two distinct blocks. Then $|H_1 \cap H_2| < t$, because any $t$-set is contained in exactly one block, and for any $x \in E$, $\{x\} \cup (H_1 \cap H_2)$ is contained in at least one $t$-set, hence in at least one block. Thus $M$ is a matroid with hyperplane family $\mathscr{H}$. By (7), the bases of $M$ are just the minimal subsets of $E$ contained in no hyperplane, that is, the $(t+1)$-subsets of $E$ contained in no hyperplane. Thus $r(M) = t+1$.

For any $t$-subset $T$ of $E$, $H = cl\ (T)$ is the unique hyperplane containing $T$. Since $H \subsetneq E$, there exists $x \in E-H$, and $\{x\} \cup T$ is contained in no hyperplane of $M$, hence $\{x\} \cup T$ is a basis, and $T$ is independent in $M$. Thus every $t$-subset of $E$ is independent. It follows that every $j$-flat of $M$, $1 \leq j \leq t-1$, is simple. In particular, since $t \geq 2$, $M$ is simple. Hence $t_M\ (0, 1, j) = j$, for $1 \leq j \leq t-1$, $t_M\ (0, 1, t) = k$, and $t_M\ (0, 1, t+1) = v$. Therefore, by definition of the $d$-sequence $d(M) = (v-k, k-t+1, 1, \ldots, 1)$.

Suppose, conversely, that $t \geq 2$ and $M$ is a simple, rank $t+1$ PMD with $d$-sequence $(v-k, k-t+1, 1, \ldots, 1)$. Then, by (42), $t_M\ (0, 1, j) = j$ for $1 \leq j \leq t-1$; that is, every $j$-flat of $M$ is simple for $1 \leq j \leq t-1$. Hence every $t$-subset $T$ of $E$ must be independent in $M$, and is therefore contained in a unique member of $\mathscr{H}$, namely $cl\ (T)$. Again using (42) and the fact that $M$ is simple, we find that the hyperplanes of $M$ have cardinality $k$, and $|E| = v$. Thus, $M = (E, \mathscr{H})$ is a $t-(v, k, 1)$ design.

(65) As a special case of (64), it follows that every BIBD with $\lambda = 1$ is a rank 3 PMD.

(66) In analogy with matroid designs, we say that a $t-(v, k, \lambda)$ design $(E, \mathscr{H})$ is *trivial* if $\mathscr{H}$ consists of all $k$-subsets of $E$.

*For* $t \geq 3$, relatively few nontrivial $t$-designs have been found, and for $T \geq 6$ no nontrivial $t$-designs are known. We shall briefly mention here all known examples of nontrivial $t-(v, k, 1)$ designs, where $t \geq 3$.

(67) For any prime power $q$ and integer $f \geq 1$, choose the finite Galois fields $GF(q)$ and $GF(q^f)$ such that $GF(q) \subseteq GF(q^f)$. Then the images of $GF(q) \cup \{\infty\}$ under the group of transformations

$$\mathscr{G} = \left\{ \frac{ax^{q^i}+b}{cx^{q^i}+d} : a, b, c, d \in GF(q^f), ad-bc \neq 0, 0 \leq i \leq f \right\}$$

are the blocks of a $3-(q^f+1, q+1, 1)$ design on the set $GF(q^f) \cup \{\infty\}$. In the case $f=2$, the $3-(q^2+1, q+1, 1)$ design of this type is called an *inversive plane of order* $q$, abbreviated by $IP(q)$. (For a more detailed treatment, see [6] and [7].)

(68) A second infinite class of 3-designs consists of the $3-(v, 4, 1)$ designs, which are known as quadruple systems. As shown in [6], necessary and sufficient conditions for the existence of a quadruple system $3-(v, 4, 1)$ are that $v \equiv 2$ or $4 \pmod 6$.

(69) The only other $t-(v, k, 1)$ designs known for $t \geq 3$ are due to Witt [13]. They have parameters $4-(11, 5, 1)$, $4-(23, 7, 1)$, $5-(12, 6, 1)$, and $5-(24, 8, 1)$, and are related to the Mathieu groups $M_{12}$ and $M_{24}$.

(70) We point out that for a sequence of form $(v-k, k-t+1, 1, \ldots, \tilde{1})$, the condition (D1) for $D$-consistency simply reduces to the requirement that the numbers $\lambda_i$ associated (22) with the corresponding $t-(v, k, 1)$ design be integers. Likewise, (D2) is simply a consequence of Fisher's inequality for this $t$-design. However, it is well-known from BIBD theory that for given parameters $t, v, k, \lambda$ the integrality of the expressions for $\lambda_i$ and Fisher's inequality are not sufficient to assure the existence of a $t-(v, k, \lambda)$ design. Hence, in particular, the $D$-consistency of a sequence does not imply that it is actually the $d$-sequence of some PMD.

A $D$-consistent sequence that is the $d$-sequence of some PMD will be called a *realizable* $D$-consistent sequence. We shall presently develop several additional criteria for $D$-consistent sequences to be realizable.

A third class of PMD's consists of the classical finite projective and affine geometries of arbitrary dimension. This class is the prototype on which PMD's are modeled.

A finite projective geometry $(E, \mathscr{L})$ is a set $E$ of points and a set $\mathscr{L}$ of lines satisfying certain well-known incidence axioms [12]. A finite projective geometry of *dimension* $n$ and *order* $s$ will be abbreviated by $PG(n, s)$, where $n \geq 2$ and $s+1 \geq 3$ is the number of points on every line, and $s^{n+1}-1/s-1$ is the total number of points. The *subspaces* of a $PG(n, s)$ are those subsets $S \subseteq E$ such that for every $x, y \in S$, the unique line containing $x$ and $y$ is a subset of $S$. The *hyperspaces* are just the maximal proper subspaces.

(71) The points and lines of a $PG(2, s)$ form a BIBD with $\lambda = 1$, namely, a $2-(s^2+s+1, s+1, 1)$ design. That is, any $PG(2, s)$ is a rank 3 PMD with $d$-sequence $(s^2, s, 1)$. Conversely, it may be shown by a simple counting argument that in any PMD with $d$-sequence $(s^2, s, 1)$, every two lines meet [5]; and hence every such PMD is a $PG(2, s)$.

(72) For $n \geq 3$, a $PG(n, s)$ exists if and only if $s$ is a prime power [12] and in this case it is isomorphic to the $PG(n, s)$ constructed as follows. Let $s = q$, where $q$ is a prime power, and let $E$ be the set of all nonzero $(n+1)$-tuples over the Galois field $GF(q)$. Let $\mathscr{I}$ be the family of all subsets of $E$ that are linearly independent over $GF(q)$. Clearly, $M = (E, \mathscr{I})$ is a rank $n+1$ matroid. Moreover, for $0 \leq j \leq n+1$, every $j$-flat of $M$ contains $q^j - 1$ elements. In particular, the points of $M$ each contain $q-1$ elements. Where $M'$ is any $(q-1)$-deflation of $M$, it follows that $M'$ is a simple, rank $n+1$ PMD with $t$-function given by $t_{M'}(0, 1, j) = q^j - 1/q - 1$ and $d$-sequence $(q^n, q^{n-1}, \ldots, q, 1)$. For any 3-flat $F$ of $M'$, $M' \times F$ has $d$-sequence $(q^2, q, 1)$ and is therefore a $PG(2, q)$. It follows easily that the point-family $E'$ and line-family $\mathscr{L}'$ of $M'$ together satisfy the projective axioms. Since every line has cardinality $q+1$, and $|E'| = q^{n+1} - 1/q - 1$, $(E', \mathscr{L}')$ is a $PG(n, q)$. The subspaces of $(E', \mathscr{L}')$ are precisely the spans of independent sets of $M'$; i.e., they are precisely the flats of $M'$.

(73) It follows from (71) and (72) that the family $E$ of points and the family $\mathscr{H}$ of hyperspaces of any $PG(n, s)$ $(E, \mathscr{L})$ are the points and hyperplanes of a rank $n+1$ PMD whose flats are the subspaces of $(E, \mathscr{L})$. In this sense, every $PG(n, s)$ is a rank $n+1$ PMD with $d$-sequence $(s^n, s^{n-1}, \ldots, s, 1)$.

(74) Suppose that $M = (E, \mathscr{H})$ is any simple PMD with $d$-sequence of form $(\ldots, s^2, s, 1)$. Then just as above we may conclude that every 3-flat of $M$ is a $PG(2, s)$, and hence that the line-family $\mathscr{L}$ of $M$ satisfies the projective axioms. Thus, $(E, \mathscr{L})$ is a $PG(h, s)$ for some integer $h$. However, this does not necessarily imply that $M$ is a $PG(h, s)$ in the sense described above. Conceivably, $M$ might be constructed so that its flats constituted a proper subcollection of the subspaces of the $PG(h, s)$ defined by the lines. On the other hand, we may observe that the total number of points of $M$ must be $s^h - 1/s - 1$. This leads to the following useful criterion, which was pointed out to the authors by Richard Wilson.

(75) If $D = (d_n, d_{n-1}, \ldots, 1)$ is a realizable $D$-consistent sequence such that $n \geq 3$ and $d_3 = s^2$, $d_2 = s$ for some $s \geq 2$, then

$$\sum_{i=1}^{n} d_i = \frac{s^h - 1}{s - 1}$$

for some positive integer $h$.

However, if we know that $d(M) = (s^n, s^{n-1}, \ldots, s, 1)$ then we may conclude that $M$ is a $PG(n, s)$.

(76) THEOREM: *For any integers* $n \geq 2$ *and* $s \geq 2$, *every simple rank* $n+1$ *PMD with* $d$*-sequence of form* $(s^n, s^{n-1}, \ldots, s, 1)$ *is a* $PG(n, s)$, *and conversely.*

PROOF: For a simple PMD $M=(E, \mathscr{H})$ with $d(M)=(s^n, s^{n-1}, \ldots, s, 1)$ and line-family $\mathscr{L}$, $(E, \mathscr{L})$ is a finite projective geometry of order $s$, by (74). Since $M$ is simple,

$$|E|=t_M(0, 1, n+1)=\sum_{i=0}^{n} s^i=\frac{s^{n+1}-1}{s-1};$$

hence $(E, \mathscr{L})$ is a $PG(n, s)$.

Let $\mathscr{H}'$ be the family of hyperspaces of $(E, \mathscr{L})$. To show that $M$ is a $PG(n, s)$, we will show that $\mathscr{H}=\mathscr{H}'$. By (73), $M'=(E, \mathscr{H}')$ is a PMD with the same $d$-sequence as $M$. Hence their $t$-functions are the same, so they contain the same number of flats. But for any flat $F$ of $M$ and distinct $x, y \in F$, $cl_M(\{x, y\}) \in \mathscr{L}$ and $cl_M(\{x, y\}) \subseteq F$; hence $F$ is a subspace of $(E, \mathscr{L})$. Therefore, every flat of $M$ is a flat of $M'$. Hence they have the same flats, and in particular $\mathscr{H}=\mathscr{H}'$. Thus $M$ is a $PG(n, s)$. The converse has already been noted in (73).

The criterion of (75) may be applied to show that certain $D$-consistent sequences are not realizable. For example, the sequence (36, 4, 2, 1) satisfies (D0), (D1), and (D2) (as may be verified directly), but $36+4+2+1=43 \neq 2^h-1$ for any integer $h$.

(77) For any $PG(n, s)$ $M=(E, \mathscr{H})$ and hyperplane $H_0$ of $M$, a matroid of form $M'=M \times (E-H_0)$, is called *an affine geometry of dimension* n *and order* s, abbreviated by $EG(n, s)$. If $F'$ is any $j$-flat of $M'$, $0 \leq j \leq n+1$, then $F=cl_M(F')$ is a $j$-flat of $M$ not contained in $H_0$. By the modular law for projective geometries ([12], Chapter 7, Theorem 10), $r(F \cap H_0)=r(F)-1$. Hence we have $|F'|=|F|-|F \cap H|=s^{j-1}$. Conversely, for any $j$-flat $F$ of $M$ not contained in $H_0$, $F'=F-H_0$ is a flat of $M'$ and $|F'|=s^{j-1}$, so, by the above, $F'$ must be a $j$-flat of $M'$. Thus, $M'$ is a rank $n+1$ PMD whose $j$-flats are precisely the sets of form $F-H_0$ for $j$-flats $F$ of $M$ not contained in $H_0$, and the $d$-sequence of $M'$ is $(s^{n-1}(s-1), s^{n-2}(s-1), \ldots, s-1, 1)$.

As in the case of projective geometries, we may also characterize affine geometries by their $d$-sequences. For this purpose we use the well-known axiomatization of affine geometries due to Lenz [8]. Let $M=(E, \mathscr{H})$ be a simple PMD with $d$-sequence

$$d(M)=(s^{n-1}(s-1), s^{n-2}(s-1), \ldots, s-1, 1)$$

for some integers $n \geq 2$ and $s \geq 2$. Where $F$ is any 3-flat of $M$, $d(M \times F)=(s^2-s, s-1, 1)$ and $t_{M \times F}(1, 2, 3)=s+1$. Thus every point is contained in $s+1$ lines, and since for every line $L$ of $M \times F$ and $x \in F-L$ there are exactly $s$ lines meeting both $x$ and $L$, there is a unique line containing $x$ and disjoint from $L$. Thus the lines of $M \times F$ decompose into parallel classes. By adjoining an ideal point for each parallel class and an ideal line we conclude that $M \times F$ is the reduction of a $PG(2, s)$ to the complement of some line; that is, it is an $EG(2, s)$.

(78) Let $\mathscr{L}$ be the family of lines of $M$, and for any $L_1, L_2 \in \mathscr{L}$ write $L_1 \| L_2$ if either $L_1=L_2$, or $L_1 \cap L_2=\emptyset$ and $r(L_1 \cup L_2)=3$. For any $L \in \mathscr{L}$ and point $x$ there is a unique line $L_x$ containing $x$ such that $L_x \| L$ – namely, $L_x=L$ if $x \in L$ and otherwise $L_x$ is the unique line containing $x$ and disjoint from $L$ in the 3-flat $cl(L \cup \{x\})$. It remains only to show that $\|$ is a transitive relation on $\mathscr{L}$ to establish Lenz's incidence axioms for the line-family of an affine geometry. Let $L_1, L_2, L_3$ be distinct lines such that $L_1 \| L_2$ and $L_1 \| L_3$. Then $K=cl(L_1 \cup L_2 \cup L_3)$ is a 4-flat and $d(M \times K)=(s^3-s^2, s^2-s, s-1, 1)$. Pick $x \in L_3$, and let $H=cl(L_2 \cup \{x\})$, $H_{12}=cl(L_1 \cup L_2)$, $H_{13}=cl(L_1 \cup L_3)$. $M'=(M \times K) \cdot (K-\{x\})$ has $d$-sequence $(s^2, s, 1)$ and $H-\{x\}$, $H_{13}-\{x\}$ are hyperplanes (2-flats) of $M'$. By (76), $M'$ is an inflation of a $PG(2, s)$, so by the modular law, $(H-\{x\}) \cap (H_{13}-\{x\})$ is a point of $M'$.

Hence $H \cap H_{13}=L$ is a line of $M \times K$ containing $x$. If $L \cap L_1=\emptyset$, then $L \| L_1$ and $x \in L$ implies $L=L_3$. Thus $L_2, L_3 \subseteq H$. If $L_2 \cap L_3 \neq \emptyset$, we would have $L_2=L_3$ (since both are parallel to $L_1$), which is contrary to hypothesis. Hence $L_2 \cap L_3=\emptyset$ and so $L_2 \| L_3$. If $L \cap L_1 \neq \emptyset$, say $L \cap L_1=\{y\}$, then $L_2 \cup \{y\} \subseteq H_{12} \cap H$ and so $H_{12}=H$. But then $L_1 \cup \{x\} \subseteq H_{13} \cap H$, whence $H=H_{13}=H_{12}$. Thus, $L_2, L_3 \subseteq H$, and since $L_2 \cap L_3=\emptyset$, we have $L_2 \| L_3$, proving the transitivity of $\|$.

It follows that $\mathscr{L}$ is the line-family of an affine geometry of order $s$ on the set $E$. Since $|E|=s^n$, $(E, \mathscr{L})$ must be an $EG(n, s)$. By (77), the subspaces of $(E, \mathscr{L})$ form a perfect matroid design $N$

with the same $d$-sequence as $M$, and every flat of $M$ is a subspace of $N$. Thus $M = N$, and we have proved the following.

(79) THEOREM: *For any integers* $n \geq 2$ *and* $s \geq 2$, *every simple rank* $n+1$ *PMD with* d*-sequence* $(s^{n-1}(s-1), s^{n-2}(s-1), \ldots, s-1, 1)$ *is an* $EG(n, s)$, *and conversely.*

### 3.7. Computation of D-Consistent Sequences

In spite of the insufficiency of the conditions (D0), (D1), and (D2) as existence criteria for PMD's, there are in fact very few sequences that satisfy even these conditions. In this section we will describe how (D0)–(D2) may be applied to determine all possible $D$-consistent sequences with a given leading term $\gamma$.

(80) Let $\gamma$ be a given positive integer with $t$ prime factors, let $D = (d_n, d_{n-1}, \ldots, d_1)$ be a $D$-consistent sequence such that $d_n = \gamma$, and let $s$ be the largest index for which $d_s = 1$. The truncated sequence $(d_n, d_{n-1}, \ldots, d_{s+1}) = D'$ is called the *nontrivial part* of $D$. By (D2), $D'$ can have at most $t$ terms. If $t$ is small, the possibilities for $D'$ will be easy to determine. The possibilities for $D$ are then obtained by extending $D'$ by sequences of "1" 's, subject to the restriction that (D1) is satisfied.

(81) Let $D = (d_n, d_{n-1}, \ldots, d_1)$ be a sequence of positive integers. A *normalized interval of* $D$ having *length* $i$ is a sequence of form

$$\left(\frac{d_{h+i}}{d_{h+1}}, \frac{d_{h+i-1}}{d_{h+1}}, \ldots, \frac{d_{h+1}}{d_{h+1}}\right),$$

where $1 \leq i \leq n$, $0 \leq h \leq n-i$. It is easily verified that the sequence $D$ is $D$-consistent if and only if every normalized interval of $D$ is $D$-consistent. This fact is often useful in checking $D$-consistency of long sequences.

(82) As an application, let us find all $D$-consistent sequences with leading term 20. From (D2) it follows that the nontrivial part of every $D$-consistent sequence with leading term 20 is one of the following: (20), (20, 2), (20, 4), or (20, 4, 2). By (59), every sequence of form $(20, 1, \ldots, 1)$ is $D$-consistent. It is easy to check that (20, 2, 1) and (20, 4, 2, 1) are not $D$-consistent; hence neither of these is the interval of a $D$-consistent sequence. The sequences (20, 4, 1) and (20, 4, 1, 1) are $D$-consistent, but (20, 4, 1, 1, 1) is not.

Thus, the only $D$-consistent sequences with leading term 20 are (20, 4, 1), (20, 4, 1, 1), and trivial sequences. By (79), $EG(2, 5)$ is the unique PMD with $d$-sequence (20, 4, 1). There is also a PMD with $d$-sequence (20, 4, 1, 1), namely, the inversive plane $IP(2, 5)$.

In the Appendix we have listed all nontrivial $D$-consistent sequences with leading terms 1–40 using the methods presented above. PMD's corresponding to the sequences are given, if known. In some cases, existence theorems for BIBD's, the Bruck-Ryser conditions for finite geometries, or Theorem (75) imply that a given $D$-consistent sequence is not realizable. Together, these conditions and $D$-consistency are sufficiently powerful that, for all $D$-consistent sequences with leading term between 1 and 40, the only realizable ones correspond to one of the three classes of PMD's previously discussed: trivioids, $t$-designs with $\lambda = 1$, and projective and affine geometries (or truncations of these). On the other hand, there is every indication that many more types of PMD's exist, probably an infinite number. For example, the sequence (48, 6, 2, 1) is $D$-consistent (as may be checked), and satisfies (75) and known BIBD existence criteria, but it does not correspond to a PMD from one of the above four classes. The existence of a PMD with this $d$-sequence has not yet been established.

### 3.8. Bounds on the Rank of Nontrivial Perfect Matroid Designs with Given Co-Circuit Cardinality

The Appendix shows that for each $N$, $1 \leq N \leq 40$ the number of nontrivial $D$-consistent sequences with leading term $N$ is finite. We shall show presently that this result is true for every

positive integer $N$, and in fact we shall be able to give sharp bounds on the rank of any nontrivial PMD with given co-circuit cardinality. As a preliminary, we shall state and prove an interesting number-theoretic result that appears to be new.

(83) For every pair $(n, m)$ of integers, $n \geq m \geq 1$, and integer $r \geq 1$, let

$$\pi_r(m, n) = \frac{n(n+1)\ldots(n+r-1)}{m(m+1)\ldots(m+r-1)} = \frac{\binom{n+r-1}{r}}{\binom{m+r-1}{r}},$$

and define the *factorial index of* m *in* n, $\beta(m, n)$, to be the maximum integer $t$ such that $\pi_r(m, n)$ is integral for $0 \leq r \leq t$. We set $\beta(m, n) = 0$ if $m \nmid n$.

(84) THEOREM. *If* m *and* n *are integers such that* $2 \leq m \leq \frac{n+1}{2}$, *then* $\beta(m, n) \leq n - 2m + 1$.

PROOF: We begin with the following identity which is a variant of Van der Monde's formula ([11], p. 9).

$$\binom{a}{t} = \sum_{r=0}^{t} (-1)^r \binom{b}{r}\binom{a+b-r}{t-r}. \tag{85}$$

Dividing both sides of (85) by $\binom{b}{t}$, and using the relation $\frac{\binom{b}{r}}{\binom{b}{t}} = \frac{\binom{t}{r}}{\binom{b-r}{t-r}}$, $0 \leq r \leq t$, we can rewrite (85) as follows:

$$\frac{\binom{a}{t}}{\binom{b}{t}} = \sum_{r=0}^{t} (-1)^r \binom{t}{r} \frac{\binom{a+b-r}{t-r}}{\binom{b-r}{t-r}}. \tag{86}$$

For given integers $n > m \geq 2$, let $0 \leq t \leq n - m$, and substitute $a = n - m$, $b = m + t - 1$ in (86). Then we obtain

$$\frac{\binom{n-m}{t}}{\binom{m+t-1}{t}} = \sum_{r=0}^{t} (-1)^r \binom{t}{r} \frac{\binom{n+t-1-r}{t-r}}{\binom{m+t-1-r}{t-r}} = \sum_{r=0}^{t} (-1)^r \binom{t}{r} \pi_{t-r}(m, n). \tag{87}$$

Hence, if $\pi_{t-r}$ is integral for $0 \leq t - r \leq t$, then $\binom{n-m}{t} \Big/ \binom{m+t-1}{t}$ is integral. If $t > 0$, it follows that $m + t - 1 \leq n - m$, so $t \leq n - 2m + 1$. If $t = 0$, this inequality also holds, since $m \leq \frac{n+1}{2}$. Hence $\beta(m, n) \leq n - 2m + 1$.

(88) For any matroid $M$, let $c(M)$ denote the minimum cardinality of a circuit of $M$.

(89) THEOREM: *If* $M = (E, \mathscr{H})$ *is a simple, nontrivial PMD with hyperplane cardinality* $k(M)$, *then*

(i) $k(M) \leq |E|/2$.

(ii) $c(M) \leq c^*(M)$.

(iii) $r(M) \leq c^*(M)$.

PROOF: Let $d(M) = D = (d_n, d_{n-1}, \ldots, d_1)$ and let $s$ be the greatest index such that $d_s = 1$. $s \leq n-2$ because $M$, and hence $D$, is non-trivial. For each $k$, $1 \leq k \leq s+1$,

$$\pi_k(d_{s+1}, d_{s+2} + d_{s+1}) = \frac{(d_{s+2} + d_{s+1})(d_{s+2} + d_{s+1} + 1) \ldots (d_{s+2} + d_{s+1} + k - 1)}{(d_{s+1})(d_{s+1} + 1) \ldots (d_{s+1} + k - 1)}$$

$$= t_M(s - k - 1, s + 1, s + 2),$$

which is an integer.

Since $\dfrac{d_{s+2} + d_{s+1} + 1}{2} \geq d_{s+1} \geq 2$, (84) implies that

$$s + 1 \leq (d_{s+1} + d_{s+2}) - 2d_{s+1} + 1$$

that is,

(90) $$s \leq d_{s+2} - d_{s+1}.$$

Now

$$\sum_{j=s+1}^{n-1} d_j = \frac{\sum_{j=s+1}^{n-1} (d_{s+2} - d_{s+1}) d_j}{d_{s+2} - d_{s+1}} \leq \frac{d_{s+1} \sum_{j=s+1}^{n-1} (d_{s+1} - d_j)}{d_{s+2} - d_{s+1}},$$

by (D2), so

(91) $$\sum_{j=s+1}^{n-1} d_j \leq \frac{d_{s+1}(d_n - d_{s+1})}{d_{s+2} - d_{s+1}}.$$

Since $M$ is simple, $k(M) = \sum_{j=1}^{n-1} d_j = s + \sum_{j=s+1}^{n-1} d_j$. Hence, by (90) and (91),

$$k(M) \leq d_{s+2} - d_{s+1} + \frac{d_{s+1}(d_n - d_{s+1})}{d_{s+2} - d_{s+1}}$$

$$= \frac{(d_{s+2} - d_{s+1})^2 - d_{s+1}^2}{d_{s+2} - d_{s+1}} + \frac{d_{s+1} d_n}{d_{s+2} - d_{s+1}}$$

$$= \frac{d_{s+2}(d_{s+2} - 2d_{s+1})}{d_{s+2} - d_{s+1}} + \frac{d_{s+1} d_n}{d_{s+2} - d_{s+1}}$$

$$\leq \frac{d_n(d_{s+2} - 2d_{s+1})}{d_{s+2} - d_{s+1}} + \frac{d_{s+1} d_n}{d_{s+2} - d_{s+1}}$$

$$= d_n,$$

the latter inequality because $d_n \geq d_{s+2}$ and $d_{s+2} \geq 2d_{s+1}$. Thus, $k(M) \leq d_n$. Since $M$ is simple, $k(M) + d_n = |E|$, hence $k(M) \leq |E|/2$, and (i) is proved.

Since $M$ is nontrivial, $k(M) > n - 1$, every hyperplane is dependent (contains a circuit). Hence $c(M) \leq k(M) \leq d_n = c^*(M)$, proving (ii). Finally, $r(M) = n \leq k(M) \leq c^*(M)$, proving (iii).

There are examples of PMD's for which (i), (ii), and (iii) of (89) are all satisfied as equalities: for example, the affine geometry $EG(3, 2)$, and the Witt design 5-(12, 6, 1). Therefore, (89) is, in a certain sense, best possible.

### 3.9. The $t$-Designs in a Perfect Matroid Design

We have already observed in (25) that the flats of a given cardinality in a simple PMD $M$ constitute the blocks of a BIBD on the points of $M$. This result can be sharpened to state that any such collection of flats forms a $t$-design for some appropriate $t$. Moreover, we shall show that not only the flats, but also the circuits, and the independent sets, of a given cardinality in a PMD form a $t$-design for an appropriate $t$. These results yield a method for deriving new $t$-designs from known ones in certain cases.

(92) Where $c(M)$ is the minimum cardinality of a circuit of the matroid $M$, it follows that $c(M)-1$ is the maximum integer $m$ such that every $m$-subset of $E$ is independent. $c(M)-1$ is called the *independence number of* $M$, $\beta(M)$.

(93) THEOREM: *Let* $\mathrm{M}=(\mathrm{E}, \mathscr{I})$ *be a rank* n *PMD with independence number* $\beta$, *where* $2 \leq \beta \leq \mathrm{n}$, *and for each* $\mathrm{j}, \beta \leq \mathrm{j} \leq \mathrm{n}$, let $\mathfrak{F}_j$ *be the set of* j*-flats of* M. *Then* $(\mathrm{E}, \mathfrak{F}_j)$ is a $\beta-(|\mathrm{E}|, \mathrm{t_M}(0,1,\mathrm{j}), \mathrm{t_M}(\beta, \mathrm{j}, \mathrm{n}))$ *design.*

PROOF: Let $j$ be given, $\beta \leq j \leq n$. Since every $\beta$-subset of $E$ is independent, every $\beta$ elements are contained in one and only one $\beta$-flat, which is contained in precisely $t_M(\beta, j, n)$ $j$-flats. Since $\beta \geq 2$, $M$ is simple, and hence every $j$-flat of $M$ contains $t_M(0,1,j)$ elements. Therefore $(E, \mathfrak{F}_j)$ is a $\beta-(|E|, t_M(0,1,j), t_M(\beta, j, n))$ design.

(94) THEOREM: *Let* $\mathrm{M}=(\mathrm{E}, \mathscr{H})$ *be a rank* n *PMD with independence number* $\beta$, $2 \leq \beta \leq \mathrm{n}$. *For each* $\mathrm{j}, \beta \leq \mathrm{j} \leq \mathrm{n}$, *the family* $\mathscr{I}_j$ *of independent sets of cardinality* j *in* M *are the blocks of a* $\beta-(|\mathrm{E}|, j, \mu_j)$ *design on the set* E, *where*

(95)
$$\mu_j = \frac{\prod_{l=\beta}^{j-1} (\mathrm{t_M}(0,1,\mathrm{n}) - \mathrm{t_M}(0,1,l))}{(\mathrm{j}-\beta)!} \quad \text{for } \mathrm{j} > \beta,$$

*and* $\mu_j = 1$ *for* $\mathrm{j}=\beta$.

PROOF: If $j=\beta$, we take $\mu_j = 1$ and the theorem obviously holds. Thus we may assume that $j > \beta$.

(96) Let $E_\beta$ be a fixed $\beta$-subset of $E$, and let $F_0 = cl\ (E_\beta)$. $E_\beta$ is independent, by definition of $\beta$. With each independent $j$-set $J$ containing $E_\beta$, we may associate an ordering $J' = (x_1, \ldots, x_{j-\beta})$ of $J - E_\beta$ and a sequence $\mathfrak{F}: F_0 \subset F_1 \subset \ldots \subset F_{j-\beta}$ of nested flats defined by $F_i = cl\ (F_0 \cup \{x_1, \ldots, x_i\})$, The sequence $\mathfrak{F}$ then has the property that $x_{i+1} \in E - F_i$ and $F_{i+1} = cl(F_i \cup \{x_{i+1}\})$ for $0 \leq i \leq j-\beta$. For each set $J$ containing $E_\beta$, such an association can be made in precisely $(j-\beta)!$ ways.

(97) Conversely, for any sequence of flats $cl\ (E_\beta) = F_0 \subset F_1 \subset \ldots \subset F_{j-\beta}$, and set $J' = \{x_1, \ldots, x_{j-\beta}\}$ such that $x_{i+1} \in E - F_i$ and $F_{i+1} = cl(F_i \cup \{x_{i+1}\})$ for $0 \leq i \leq j-\beta$, the set $J = J' \cup E_\beta$ is an independent set of rank $j$. If this were not so, let $k$ be the least integer such that $E_\beta \cup \{x_1, \ldots, x_k\}$ were dependent. But by (5), $F_{k-1} = cl(E_\beta \cup \{x_1, \ldots, x_{k-1}\})$, and hence we would have $x_k \in F_{k-1}$, which is contrary to assumption. Therefore $J$ is an independent set of rank $j$ containing $E_\beta$.

A sequence of flats and ordered independent set as in (97) can be chosen in precisely

$$\prod_{l=\beta}^{j-1} (t_M(0,1,n) - t_M(0,1,l))$$

distinct ways, and each corresponding independent $j$-set containing $E_\beta$ is thereby counted $(j-\beta)!$ times, by (96). Thus $E_\beta$ is contained in precisely

$$\mu_j = \frac{\prod_{l=\beta}^{j-1} (t_M(0,1,n) - t_M(0,1,l))}{(j-\beta)!}$$

independent $j$-sets.

(98) THEOREM: *Let* $\mathrm{M}=(\mathrm{E}, \mathscr{H})$ *be a simple, rank* n *PMD with independent number* $\beta$. *For each*

$j, \beta < j \leq n+1$, *the family* $\mathscr{C}_j$ *of cardinality-j circuits of* M *are the blocks of a* $\beta-(|E|, j, \eta_j)$ *design on the set* E, *where*

$$\eta_j = \mu_{j-1}\left[\frac{t_M(0,1,n) + \sum_{i=1}^{j-2}(-1)^{j-i-1}\binom{j}{i}t_M(0,1,i)}{j-\beta}\right] - \mu_j, \tag{99}$$

*and* $\mu_j$ *is as defined in* (95) *for* $\beta < j \leq n$, *and* $\mu_{n+1} = 0$.

PROOF: For each $j, 2 \leq j \leq n+1$, let

$$f(j) = t_M(0,1,n) + \sum_{i=1}^{j-2}(-1)^{j-i-1}\binom{j}{i}t_M(0,1,i).$$

We shall first establish (98) for the case $j = n+1$.

(100) Let $J$ be a basis of $M$, and for each $i$, $1 \leq i \leq n$, let $P_i$ be the set of all pairs $(I_i, x)$ such that $I_i$ is an $i$-subset of $J$ and $x \in cl(I_i)$, where $cl(I_i)$ is the unique $i$-flat containing $I_i$. Since $M$ is simple, $t_M(0,1,i)$ represents the cardinality of any $i$-flat, $1 \leq i \leq n$. Thus we have $|P_i| = \binom{|J|}{i} t_M(0,1,i) = \binom{n}{i} t_M(0,1,i)$. Let $P = \bigcup_{i=1}^{n} P_i$, and to each pair $(I, x) \in P$ assign the coefficient $(-1)^{n-i}$ if $(I, x) \in P_i$. Thus $f(n+1)$ is the sum of the coefficients of the members of $P$.

For any $x \in E$, define $F(x)$ to be the sum of the coefficients of pairs in $P$ containing $x$. If $x \in J$, $x$ is contained in the closure of $\binom{n-1}{i-1}$ $i$-subsets of $J$, $1 \leq i \leq n$. Thus

$$F(x) = \sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1} = 0 \text{ for } x \in J. \tag{101}$$

If $x \in E-J$, then since $J$ is a maximal independent set, $J \cup \{x\}$ contains a circuit $C_x$, containing $x$, and by (1), it is unique. Let $|C_x| = j$; clearly, $j \leq n+1$. Now $(I, x) \in P$ if and only if $C_x \subseteq I \cup \{x\}$. Thus $x$ is contained in the closures of $\binom{n-j+1}{i-j+1}$ $i$-subsets of $J$ for $j-1 \leq i \leq n$, and $x$ is contained in the closures of no $i$-subsets of $j$ for $0 \leq i \leq j-2$. Hence $F(x) = \sum_{i=j-1}^{n}(-1)^{n-i}\binom{n-j+1}{i-j+1}$ in this case. Setting $n-j+1 = k$ and $i-j+1 = l$, we have that

$$\sum_{i=j-1}^{n}(-1)^{n-i}\binom{n-j+1}{i-j+1} = \sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l} = \begin{cases} 0, \text{ for } k > 0. \\ 1, \text{ for } k = 0, \text{ i.e., for } j = n+1. \end{cases}$$

Thus $F(x) = 1$ if and only if $x \in E-J$ and $|C_x| = n+1$, i.e., if and only if $J \cup \{x\}$ is a circuit. Since $f(n+1) = \sum_{x \in E} F(x)$, $f(n+1)$ is precisely the number of circuits of cardinality $n+1$ containing $J$.

By (94), every $\beta$-subset of $E$ is contained in exactly $\mu_n$ bases of $M$, where $\mu_n$ is as in (95). Further, every circuit $C$ of cardinality $n+1$ contains $n+1-\beta$ bases of $M$ containing a given $\beta$ subset of $C$. Hence the number of cardinality $-(n+1)$ circuits of $M$ containing any given $\beta$-subset of $E$ is exactly

$$\eta_{n+1} = \frac{f(n+1)}{n+1-\beta} = \mu_n \frac{t_M(0,1,n) + \sum_{i=1}^{n-1}(-1)^{n-i}\binom{n}{i}t_M(0,1,i)}{n+1-\beta}. \tag{102}$$

(103) For an arbitrary $j, \beta < j \leq n+1$, we consider $M^{(j-1)}$, the $(j-1)$-truncation of $M$. The circuits of

$M^{(j-1)}$ are evidently the circuits of $M$ having cardinality less than or equal to $j$, together with the $j$-subsets of $E$ that are independent in $M$. By (37), $M^{(j-1)}$ has rank $j-1$ and its $t$-function $t(i, j, k)$ is determined by

$$t(0, 1, i) = t_M(0, 1, i), \quad 0 \leq i \leq j-2,$$

$$t(0, 1, j-1) = t_M(0, 1, n).$$

Hence (102) applied to $M^{(j-1)}$ implies that every $\beta$-subset of $E$ is contained in precisely

$$\mu_{j-1}\left[\frac{t_M(0, 1, n) + \sum_{i=1}^{j-2} (-1)^{j-i-1}\binom{j-1}{i} t_M(0, 1, i)}{j-\beta}\right] = \frac{\mu_{j-1} f(j)}{j-\beta}$$

cardinality-$j$ circuits of $M^{(j-1)}$. However, $\mu_j$ of these circuits are cardinality-$j$ independent sets of $M$, by (103). Thus every $\beta$-subset of $E$ is contained in precisely

$$\eta_j = \frac{\mu_{j-1} f(j)}{j-\beta} - \mu_j$$

cardinality-$j$ circuits of $M$.

(104) Let us observe that we may have $\eta_j = 0$ for some integers $j$ in (99); in this case $\mathscr{C}_j = \varnothing$ and $(E, \mathscr{C}_j)$ is said to be a *vacuous* $\beta$-design.

### 3.10. Constructions of New $t$-Designs

Before applying the results of (94) and (98) to the construction of new $t$-designs, we shall point out several very natural ways of obtaining new $t$-designs from given ones.

The following result generalizes a well-known theorem for BIBD's.

(105) THEOREM: *Let $(E, \mathscr{H})$ be a* $\mathrm{t}-(\mathrm{v}, \mathrm{k}, \lambda)$ *design and let $\mathscr{H}'$ be the system consisting of the complements in* E *of the blocks of $\mathscr{H}$ (including repetitions). Then* $(\mathrm{E}, \mathscr{H}')$ *is a* $\mathrm{t}-(\mathrm{v}, \mathrm{v}-\mathrm{k}, \lambda')$ *design, called the* complementary design *to* M, *and*

$$\lambda' = \lambda \frac{\binom{\mathrm{v}-\mathrm{k}}{\mathrm{t}}}{\binom{\mathrm{k}}{\mathrm{t}}}.$$

PROOF: For integer $j$, $0 \leq j \leq t$, let $\lambda_j$ be the number of blocks containing any fixed $j$-subset of $E$. For a given $j$-set $E_j$ and $t$-set $E_t$, where $E_j \subseteq E_t \subseteq E$, let $\eta_j$ be the number of blocks $H$ such that $H \cap E_t = E_j$. Then $\eta_t = \lambda$, and recursively,

$$\eta_j = \lambda_j - \sum_{i=1}^{t-j}\binom{t-j}{i}\eta_{j+i}$$

is independent of the particular sets $E_j$ and $E_t$. It follows that $(E, \mathscr{H}')$ is a $t-(v, v-k, \eta_0)$ design Hence

$$\lambda\frac{\binom{v}{t}}{\binom{k}{t}} = |\mathscr{H}| = |\mathscr{H}'| = \eta_0\frac{\binom{v}{t}}{\binom{v-k}{t}}.$$

so

$$\eta_0 = \lambda \frac{\binom{v-k}{t}}{\binom{k}{t}}.$$

(106) Let $(E, \mathscr{H})$ be a $t-(v, k, \lambda)$ design such that $\mathscr{H}$ contains no repeated blocks and let $\mathscr{H}'$ be the family of $k$-subsets of $E$ not contained in $\mathscr{H}$. Clearly, $(E, \mathscr{H}')$ is a $t-(v, k, \lambda')$ design, where $\lambda' = \binom{v-t}{k-t} - \lambda$. $(E, \mathscr{H}')$ will be called the *opposite design to* $M$. This obvious construction seems to have passed unnoticed in the literature on $t$-designs.

(107) Let $(E, \mathscr{H})$ be a $t-(v, k, \lambda)$ design and suppose that for some integers $l < k$ and $\lambda'$ there exists a $t-(k, l, \lambda')$ design $(E', \mathscr{H}')$. For each $H \in \mathscr{H}$, let $(H, \mathscr{H}'_H)$ be any design on the set $H$ that is isomorphic to $(E', \mathscr{H}')$, and let $\mathscr{H}''$ be the union (with repetitions) of the systems $\mathscr{H}'_H, H \in \mathscr{H}$. Then it is straightforward to verify that $(E', \mathscr{H}'')$ is a $t-(v, l, \lambda\lambda')$ design, called a *composition of* $(E, \mathscr{H})$ *with* $(E', \mathscr{H}')$. (This construction for BIBD's was pointed out by Hanani [7].)

Now let $M = (E, \mathscr{H})$ be a given $t-(v, k, 1)$ design.

(108) By (64), $r(M) = t+1$ and $\beta(M) = t$. Hence every circuit of $M$ has cardinality $t+1$ or $t+2$. In this case, (94) and (98) have particularly simple interpretations.

(109) The family $\mathscr{C}_{t+1}$ of circuits of cardinality $(t+1)$ consists of precisely the $(t+1)$ subsets of the members of $\mathscr{H}$. Indeed, for any $H \in \mathscr{H}$, $r(H) = r(M) - 1 = t$, so every $(t+1)$-subset of $H$ is dependent, and in fact minimal dependent, since $\beta(M) = t$. Conversely every circuit $\mathscr{C}$ of cardinality $t+1$ is contained in a hyperplane $H \in \mathscr{H}$, since $r(\mathscr{C}) = t$. Thus $(E, \mathscr{C}_{t+1})$ is a composition of $(E, \mathscr{H})$ with a $\sigma(1, t+1, k)$; i.e., $(E, \mathscr{C}_{t+1})$ is a $t-(v, t+1, k-t)$ design.

(110) The family $\beta$ of bases of $M$ is evidently precisely the collection of $(t+1)$-subsets of $E$ that are not contained in $\mathscr{C}_{t+1}$. Thus $(E, \beta)$ is a $t-(v, t+1, v-k)$ design, the opposite to $(E, \mathscr{C}_{t+1})$.

(111) Finally, the family $\mathscr{C}_{t+2}$ of cardinality $(t+2)$ circuits of $M$ consists of precisely those $(t+2)$-subsets $S$ of $E$ such that every $(t+1)$-subset of $S$ is a basis of $M$. Using (162), we find that $(E, \mathscr{C}_{t+2})$ is a

$$t-\left(v, t+2, \frac{(v-k)\,(v-(t-1)\,(k-t+1))}{2}\right)$$

design, which we shall call the *circuit design* of $M$.

The construction of (111) applies to any $t-(v, k, 1)$ design, and is apparently new. Table 1 shows the parameters of the known $t-(v, k, 1)$ designs with $t \geq 3$, and the associated circuit design parameters.

TABLE 1. *Parameters of known* t-(v, k, 1) *designs,* (t ≥ 3) *and associated circuit designs*

| $t$-Design parameters | Circuit design parameters |
|---|---|
| $3-(6u+2, 4, 1)$, $u \geq 0$ | $3-(6u+2, 5, 6(u-1)(3u-1))$ |
| $3-(6u+4, 4, 1)$, $u \geq 0$ | $3-(6u+4, 5, 6u(3u-2))$ |
| $3-(q^n+1, q+1, 1)$, $q$ a prime. | $3-\left(q^n+1, 5, \frac{(q^n-q)(q^n-4q+5)}{2}\right)$ |
| $4-(11, 5, 1)$ | $4-(11, 6, 3)$ |
| $4-(23, 7, 1)$ | $4-(23, 6, 24)$ |
| $5-(12, 6, 1)$ | vacuous, $\lambda = 0$ |
| $5-(24, 8, 1)$ | vacuous, $\lambda = 0$ |

### 3.11. Self-Dual PMD's

We close our discussion of PMD's by developing an intriguing unsolved problem in the theory of $t$-designs.

(112) A matroid $M=(E, \mathscr{H})$ is said to be *self-dual* if $M^*=M$; i.e., if the hyperplanes of $M^*$ are identical with the hyperplanes of $M$.

(113) If $M$ is self-dual, then in particular the bases of $M$ and $M^*$ are identical, so for every basis $B$ of $M$, $E-B$ is also a basis of $M$. Hence $|E-B|=|B|$, so $r(M)=\frac{|E|}{2}$.

(114) THEOREM: M *is a simple, nontrivial, self-dual PMD if and only if* M *is a* (k−1)−(2k, k, 1) *design,* k > 2. *Moreover, such an* M *can exist only if* k+1 *is an odd prime.*

PROOF: Let $M=(E, \mathscr{H})$ be a simple, nontrivial, self-dual PMD and let $k(M)=k$. By (89), $k \leq \frac{|E|}{2}$. Further, $\frac{|E|}{2}=r(M)$, by (113). But $k=|H| \geq r(H)=r(M)-1$ for every $H \in \mathscr{H}$, so $\frac{|E|}{2}-1 \leq k \leq \frac{|E|}{2}$. If $k=\frac{|E|}{2}-1$, then evidently every hyperplane is a rank $r(M)-1$ independent set. But then every subset of cardinality $r(M)$ is a basis, so $M$ is trivial, contrary to hypothesis. Hence $k=r(M)=\frac{|E|}{2}$, and every hyperplane contains a circuit. But the circuits are identical with the co-circuits, hence they all have cardinality $|E|-k=k$. Thus every hyperplane is a circuit, and conversely.

Since the circuits all have cardinality $k$, every $(k-1)$-subset of $E$ is independent. Thus, for $0 \leq j \leq k-2$, the $j$-flats of $M$ are all simple. Thus, $M$ is a PMD with $d$-sequence $(k, 2, 1, 1, \ldots, 1)$ containing $k-2$ "1"'s. It follows from (64) that $M$ is a $(k-1)-(2k, k, 1)$ design.

Now $k>2$, for otherwise $M$ would be nonsimple. Suppose that $k+1$ is not a prime, and let $p$ be a prime divisor of $k+1$, where $1<p<k+1$. By (42),

$$t_M(k-p-1, k-1, k)=\frac{(k+2+(p-2))(k+2+(p-3)) \ \ldots \ (k+2)}{(2+(p-2))(2+(p-3)) \ \ldots \ 2}$$

$$=\frac{(k+p)(k+p-1) \ \ldots \ (k+2)}{p(p-1) \ \ldots \ 2} \tag{115}$$

must be an integer. However, since $p$ divides $(k+1)$, $p$ does not divide any integer strictly between $k+1$ and $k+1+p$. Thus $p$ does not divide the numerator of (115), which gives a contradiction.

(116) Suppose, conversely, that $M=(E, \mathscr{H})$ is a $D_{k-1}(2k, k, 1)$, $k>2$. Then $\beta(M)=k-1 \geq 2$, so $M$ is simple. $M$ is nontrivial, because every $(k-1)$-subset of $E$ is contained in more than one $k$-subset when $|E|=2k \geq 6$. Finally, it has been observed by Mendelsohn [9] that for every $(k-1)-(2k, k, 1)$ design $(V, \mathscr{W})$, $W \in \mathscr{W}$ implies $E-W \in \mathscr{W}$. It follows that for every basis $B$ of $M$, $E-B$ is also a basis; hence $M=M^*$. Therefore, $M$ is a simple, nontrivial, self-dual PMD and, by the first part of the proof, $k+1$ is an odd prime.

(117) Corresponding to the primes 3, 5, and 7 are the self-dual PMD's $D_1(4, 2, 1)$, $D_3(8, 4, 1)$, and $D_5(12, 6, 1)$, respectively. Moreover, these are the unique simple, nontrivial self-dual PMD's for these orders. The existence of higher order examples is an unsolved problem.

## 4. Constructions of Matroid Designs and an Analysis of the Witt Design

In this concluding section we shall turn our attention to the general subject of matroid designs. In a matroid design $M$ we only demand that all of the hyperplanes have the same cardinality, which we denote as before by $k(M)$. In particular, the points of a matroid design may have different cardinalities, which allows more structural freedom than is possible in PMD's. However, *simple* matroid designs are virtually as difficult to construct as PMD's. In fact, it might even be imagined

that equicardinality of the points and of the hyperplanes of a matroid forces it to be a PMD. However, we shall show that this is not the case. In fact, we shall presently demonstrate a simple matroid design that is a 5-design, but not a perfect matroid design. We shall also present several basic constructions for matroid designs to illustrate the variety and complexity of these interesting structures.

The basis for our analysis is the knowledge that there exists a $t$-design with parameters 5-(24, 8, 1). This extraordinary design was discovered by Witt [13] and is constructed by considering the action of the Mathie permutation group $M_{24}$ on a certain 8-subset of a 24-element permutation set. He also showed it to be the unique $t$-design with the given parameters, up to isomorphism. Unfortunately, its presentation as such gives little insight into its structure. However, regarding it as a matroid, and using only the knowledge of its parameters, we can derive considerable information about its properties. Simultaneously, we shall be led to discover a number of interesting and important matroid designs.

Let us denote Witt's 5-(24, 8, 1) design by $W_{24}$. By (64), $W_{24}$ is a rank 6 PMD with hyperplane family $\mathscr{H}$, and its $d$-sequence is (16, 4, 1, 1, 1, 1).

(118) THEOREM: *The family $\mathscr{C}$ of circuits of $W_{24} = (E, \mathscr{H})$ is precisely the set of 6-subsets of members of $\mathscr{H}$, and $(E, \mathscr{C})$ is a 5-(24, 6, 3) design.*

PROOF: By (108) every circuit of $W_{24}$ has cardinality 6 or 7. Let $\mathscr{C}_7$ be the set of circuits of cardinality 7. Then, substituting $v=24$, $k=8$, and $t=5$ into the expression for $\lambda$ in (111), we have that $(E, \mathscr{C}_7)$ is a 5-(24, 7, 0) design; i.e., $\mathscr{C}_7=\emptyset$. Hence every circuit of $W_{24}$ has cardinality 6, and it follows from (109) that the circuits of $W_{24}$ are precisely the 6-subsets of members of $\mathscr{H}$. Therefore $(E, \mathscr{C})$ is the composition of $(E, \mathscr{H})$ with a $\sigma(1, 6, 8)$, so $(E, \mathscr{C})$ is a 5-(24, 6, 3) design.

(119) THEOREM: *The dual of $W_{24}$, $W_{24}^*$, is a simple rank 18 matroid design that is a 5-design but not a PMD.*

PROOF: Let $W_{24}=(E, \mathscr{H})$, with circuit family $\mathscr{C}$. By (8), the set $\mathscr{H}^*$ of hyperplanes of $W_{24}^*$ is precisely the set of complements in $E$ of the circuits of $W_{24}$; i.e., $(E, \mathscr{H}^*)$ is the complementary design to $(E, \mathscr{C})$. By (118), $(E, \mathscr{C})$ is a 5-(24, 6, 3) design, so it follows from (105) that $(E, \mathscr{H}^*)$ is a 5-(24, 18, $\binom{18}{5}/2$) design. Moreover, $W_{24}^*$ is a matroid design, $k(W_{24}^*)=18$, and $r(W_{24}^*)=|E|-r(W_{24})=18$. The circuits of $W_{24}^*$ are the complements of hyperplanes of $W_{24}$; hence they all have cardinality 16. In particular, $W_{24}^*$ is simple. Hence $W_{24}^*$ is a simple rank 18 matroid design that is a 5-design.

If $W_{24}^*$ were a PMD, then by (118) it would either be a trivioid (which is clearly not the case), or it would satisfy $k(W_{24}^*) \leq |E|/2$, which is also not the case.

Using (119) and the basic properties of matroid flats, we can describe the flats of $W_{24}^*$ exactly.

(120) THEOREM: (i) *For $0 \leq j \leq 14$, the $j$-flats of $W_{24}^* = (E, \mathscr{H}^*)$ are just the $j$-subsets of $E$.*

(ii) *The 15-flats consist of all circuits, and all 15-subsets of $E$ not contained in a circuit.*

(iii) *The 16-flats consist of all subsets of form $C \cup \{x\}$, where $C$ is a circuit and $x \in E - C$, and all 16-subsets of $E$ no 15 of which are contained in a circuit.*

(iv) *The 17-flats (hyperplanes) consist precisely of the sets of form $C \cup \{x, y\}$, where $C$ is a circuit and $x, y \in E - C$, $x \neq y$.*

PROOF: For $0 \leq j \leq 14$, let $T$ be any $j$-subset of $E$. Since the circuits of $W_{24}^*$ all have cardinality 16, $T$ is independent, and $cl(T)=T$ by definition of closure. Hence, for $0 \leq j \leq 14$, the $j$-flats or $W_{24}^*$ are just the $j$-subsets of $E$, and (i) is proved.

(121) Let $F^{15}$ be any 15-flat of $W_{24}^*$, and let $H$ be a hyperplane containing $F^{15}$. Since $r(H)=r(F^{15})+2$, $|H| \geq |F^{15}|+2$, and since $|H|=18$, we have $|F^{15}| \leq 16$. If $|F^{15}|=16$, $F^{15}$ must be dependent, so $F^{15}$ is a circuit. Otherwise, $|F^{15}|=15$, and since $F^{15}$ is closed, $F^{15}$ is a set of 15 elements contained in no circuit. Conversely, every circuit $C$ has rank 15, hence is contained in some 15-flat $F^{15}$; since $|F^{15}| \leq 16$, $C=F^{15}$. If $T$ is a 15-subset of $E$ contained in no circuit, then $r(T)=15$ and $cl(T)=T$, so $T$ is a 15-flat. This proves (ii).

Let $F^{16}$ be an arbitrary 16-flat of $W_{24}^*$. Using an argument like that of (121), we may show that

(122) $$16 \leq |F^{16}| \leq 17.$$

If $|F^{16}|=17$, $F^{16}$ must be dependent; hence it is of form $C \cup \{x\}$ for some circuit $C$ and $x \in E-C$. If $|F^{16}|=16$, then $cl(F^{16})=F^{16}$, so $F^{16} \cup \{x\}$ contains no circuit for any $x \in E$; hence no 15-subset of $F^{16}$ is contained in a circuit. Conversely, any set $C \cup \{x\}$, where $C$ is a circuit and $x \in E-C$, has rank 16 (since $C$ is a 15-flat) and cardinality 17, so, by (122), $C \cup \{x\}$ is a 16-flat. If $T$ is a 16-subset of $E$ such that no 15-subset of $T$ is contained in a circuit, then $cl(T)=T$ and $T$ is independent, so $T$ is a 16-flat. This proves (iii).

The characterization of the hyperplanes (iv) follows immediately from (118) and the fact that the hyperplanes and circuits of $W_{24}^*$ are the complements, respectively, of the circuits and hyperplanes of $W_{24}$.

Using matroid theory alone, the possible structures of all contractions $W_{24}^* \cdot (E-F)$, where $F$ is a flat of $W_{24}^*$, can be identified. As a way of introducing both concrete examples and general constructions of new matroid designs, we shall identify all the structures of $W_{24}^* \cdot (E-F)$ for $r(F)=14$ and $r(F)=15$.

We begin by describing a general matroid construction due to U.S.R. Murty. Let $M=(E, \mathscr{H})$ be a rank $n$ matroid (not necessarily a matroid design) and let $\mathfrak{F}_{n-2}$ be the set of $(n-2)$-flats of $M$. For a given positive integer $m$, let $P(m)$ denote the matroid consisting of a single point on $m$ elements (we shall identify $P(m)$ with the set of $m$ elements itself), and choose $P(m)$ disjoint from $E$. Then it may easily be verified that $\mathscr{H}'=\mathscr{H} \cup \{P(m) \cup F : F \in \mathfrak{F}_{n-2}\}$ satisfies (H1) and (H2); hence $\mathscr{H}'$ is the family of hyperplanes of a matroid on the set $E'=E \cup P(m)$. $(E', \mathscr{H}')$ is called the *one-point extension of $M$ by $P(m)$*, and denoted by $M \oplus P(m)$.

(123) Suppose now that $M$ is a matroid design, $k(M)=k$, and suppose further that the $(n-2)$-flats of $M$ all have cardinality $h<k$. Then clearly $M \oplus P(k-h)$ will be a matroid design with hyperplane size $k$.

We then have the following structural theorem about $W_{24}^*$. (For the proofs of this and the next theorem, the reader is referred to [15].)

(124) THEOREM: *For any 15-flat* F *of* $W_{24}^*$, $W_{24}^* \cdot (E-F)$ *has rank 3 and is either a* $\sigma(1, 2, 8)$ *or a* $PG(2,2) \oplus P(2)$, *or an* $EG(2,3)$.

$PG(2, 2) \oplus P(2)$ is shown in figure 1. In this and subsequent matroid diagrams, the simple points will be represented by solid nodes, and points with $m>1$ elements will be represented by a hollow node containing the number $m$. 2-flats will be represented by lines, and 3-flats by planes, where convenient.

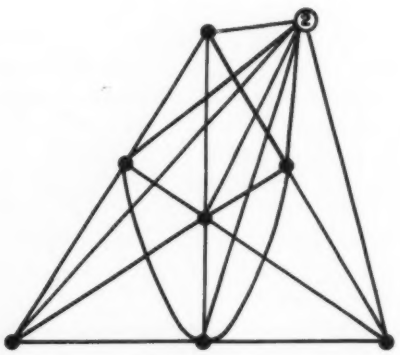


FIGURE 1. *The matroid design* $PG(2,2) \oplus P(2)$.

The following matroid design arises as a rank 4 contraction of $W_{24}^*$. It is the smallest example of a simple matroid design that is not a PMD.

(125) Let $L_1$, $L_2$ be disjoint sets of three elements each, and let $Q$ be the set of all nine pairs $\{x, y\}$ such that $x \in L_1$, $y \in L_2$. Let $Q$ be partitioned into three sets $P_1$, $P_2$, $P_3$, each containing three pairs, such that the pairs in each $P_i$ are mutually disjoint, $1 \leq i \leq 3$. Now let $A$ be a set of four

elements disjoint from $L_1 \cup L_2$, and let $Q'$ be the set of six distinct pairs $\{x, y\}$ in $A$. Let $P'_1, P'_2, P'_3$ be a partitioning of $Q'$ into three sets, each containing two disjoint pairs of $Q'$. We define

$$E = A \cup L_1 \cup L_2;$$

$$Y_i = \{L_i \cup \{y\}: y \in E - L_i\}, \qquad \text{for } i = 1, 2,$$

$$Z_j = \{q \cup q': q \in P_j, \quad q' \in P'_j\}, \qquad j = 1, 2, 3,$$

and

$$\mathscr{H} = \left(\bigcup_{j=1}^{3} Z_j\right) \cup \left(\bigcup_{i=1}^{2} Y_i\right) \cup \{A\}.$$

It is straightforward to verify that $(E, \mathscr{H})$ satisfies (H1) and (H2), hence is a matroid. We shall denote any matroid of this form by $\Psi$. By construction, $\Psi$ is a matroid design, and $k(\Psi) = r(\Psi) = 4$. The lines of $\Psi$ are precisely the sets of form $L = H_1 \cap H_2$ such that $|H_1 \cap H_2| \geq 2$ and $H_1$, $H_2$ are distinct members of $\mathscr{H}$. In particular, $L_1$ and $L_2$ are lines of $\Psi$, and every other line of $\Psi$ is simple. Therefore $\Psi$ is not a PMD. Finally, for every two distinct elements $x, y \in E$, there is a hyperplane containing $x$ but not $y$, so $\Psi$ itself is simple.

Representative lines and planes of $\Psi$ are shown in figure 2.

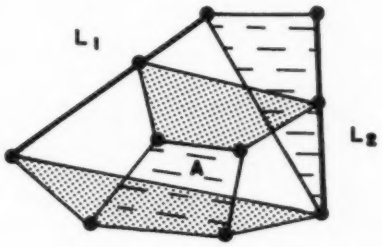


FIGURE 2. *The matroid design* $\Psi$.

(126) THEOREM: *For any 14-subset* F *of* $W_{24}^*$, $M = W_{24}^* \cdot (E - F)$ *is either*

(i) *an* EG(3, 2) $\oplus$ *P*(2)

or (ii) $\Psi$

or (iii) *an* IP(3).

Moreover, each of these matroids actually occurs for some choice of $F$.

The complete analysis of the higher-rank contractions of $W_{24}^*$ becomes quite tedious, and we shall forego it here. However, to illustrate the intricacy of structure possible in matroid designs, we shall present one example of a simple rank 5 matroid design that arises as a contraction $W_{24}^* \cdot (E - F)$, where $F$ is a 13-flat of $W_{24}^*$ contained in a circuit. We shall call this matroid $\Phi = (E', \mathscr{H}')$, and we describe it as follows.

(127) $|E'| = 11$, and all points of $\Phi$ are simple. $\Phi$ has one line of cardinality 3, $L = \{a, b, c\}$, and all other lines are simple. The remaining eight points of $\Phi$ are divided into two disjoint 3-flats having 4 elements each: $F_1^3$: $\{\alpha, \beta, \gamma, \delta\}$ and $F_2^3 = \{1, 2, 3, 4\}$. The sets of form $\{x\} \cup L$, where $x \in E' - L$, are also 3-flats of $M$. All other 3-flats are simple.

(128) $k(\Phi) = 5$ and for each $x \in L$, $\Phi \cdot (E - \{x\})$ is an $EG(3, 2) \oplus P(2)$, where $P(2) = L - \{x\}$. In other words, for each $x \in L$, the family $\mathscr{H}'_x$ of hyperplanes containing $x$ but not $L$, when restricted to the set $E - L$, form an $EG(3, 2)$ on $E - L$. In particular, $F_i^3 \cup \{x\} \in \mathscr{H}'_x$ for $i = 1, 2$ and each $x \in L$.

Let $\mathscr{H}''_x = \{H - \{x\}: H \in \mathscr{H}'_x\}$ for each $x$, $x = a, b, c$. Then $F_i^3 \in H''_x$ for each $x \in L$ and $i = 1, 2$. Moreover, $F_1^3$ and $F_2^3$ are the only sets common to any two of the families $\mathscr{H}''_x$, because they are the only 3-flats of $\Phi$ having cardinality 4 *and* contained in $E - L$.

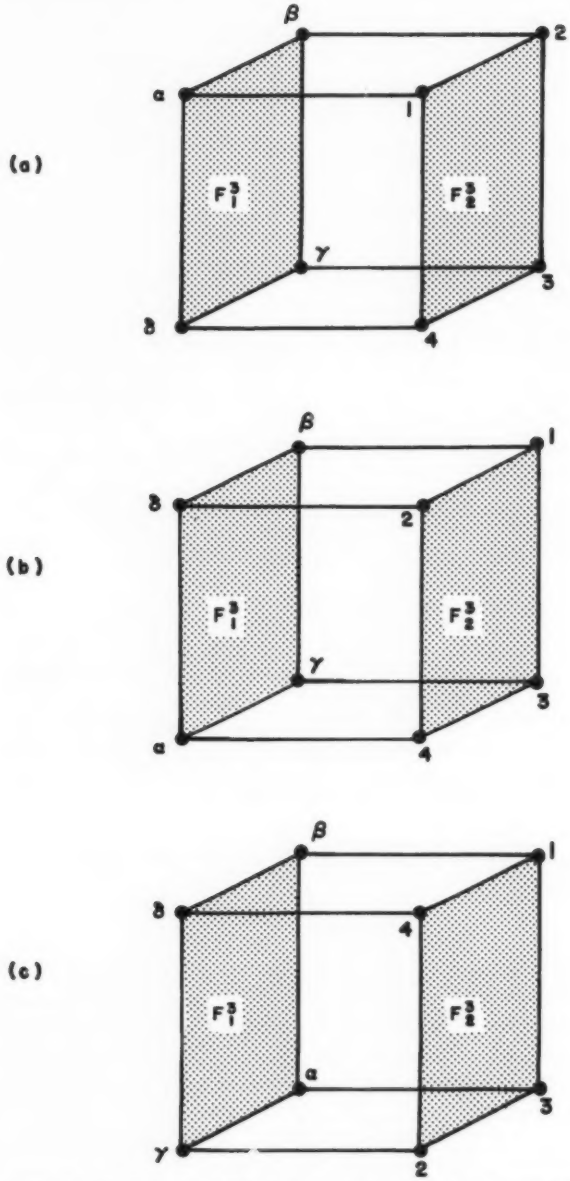


FIGURE 3. *Three copies of* EG (3, 2) *containing exactly two parallel hyperplanes in common.*

Thus, for $x=a, b, c$, the pairs $(E-L, \mathscr{H}''_x)$ constitute three copies of $EG(3,2)$ on the set $E-L$, each containing $F_1^3=\{\alpha, \beta, \gamma, \delta\}$ and $F_2^3=\{1, 2, 3, 4\}$ as their only common hyperplanes. Up to isomorphism there is only one way in which three sets of hyperplanes of an $EG(3,2)$ may be so chosen. Three such sets are listed below, and the corresponding hyperplanes are diagrammed in Figure 3.

| $(a)$ | $(b)$ | $(c)$ |
|---|---|---|
| $\{\alpha,\beta,\gamma,\delta\}$ | $\{\alpha,\beta,\gamma,\delta\}$ | $\{\alpha,\beta,\gamma,\delta\}$ |
| $\{1,2,3,4\}$ | $\{1,2,3,4\}$ | $\{1,2,3,4\}$ |
| $\{1,2,\alpha,\beta\}$ | $\{1,2,\alpha,\gamma\}$ | $\{1,2,\alpha,\delta\}$ |
| $\{1,2,\gamma,\delta\}$ | $\{1,2,\beta,\delta\}$ | $\{1,2,\beta,\gamma\}$ |
| $\{1,3,\alpha,\gamma\}$ | $\{1,3,\alpha,\delta\}$ | $\{1,3,\alpha,\beta\}$ |
| $\{1,3,\beta,\delta\}$ | $\{1,3,\beta,\gamma\}$ | $\{1,3,\gamma,\delta\}$ |
| $\{1,4,\alpha,\delta\}$ | $\{1,4,\alpha,\beta\}$ | $\{1,4,\alpha,\gamma\}$ |
| $\{1,4,\beta,\gamma\}$ | $\{1,4,\gamma,\delta\}$ | $\{1,4,\beta,\delta\}$ |
| $\{2,3,\alpha,\delta\}$ | $\{2,3,\alpha,\beta\}$ | $\{2,3,\alpha,\gamma\}$ |
| $\{2,3,\beta,\gamma\}$ | $\{2,3,\gamma,\delta\}$ | $\{2,3,\beta,\delta\}$ |
| $\{2,4,\alpha,\gamma\}$ | $\{2,4,\alpha,\delta\}$ | $\{2,4,\alpha,\beta\}$ |
| $\{2,4,\beta,\delta\}$ | $\{2,4,\beta,\gamma\}$ | $\{2,4,\gamma,\delta\}$ |
| $\{3,4,\alpha,\beta\}$ | $\{3,4,\alpha,\gamma\}$ | $\{3,4,\alpha,\delta\}$ |
| $\{3,4,\gamma,\delta\}$ | $\{3,4,\beta,\delta\}$ | $\{3,4,\beta,\gamma\}$ |

For each $x$, $x=a, b, c$, adjoin the element $x$ to each of the sets in list $(x)$, and let $\mathscr{S}_1$ be the resulting collection of 5-subsets of $E'$. Further, let $\mathscr{S}_2=\{\{x\}\cup F_i^3: x\in F_j^3, i\neq j\}$ and $\mathscr{S}_3=\{\{x, y\}\cup L: x, y\in E'-L$ and $x\neq y\}$. We claim that $\mathscr{H}'=\mathscr{S}_1\cup\mathscr{S}_2\cup\mathscr{S}_3$ is the hyperplane family of the matroid $\Phi$ having points, lines and 3-flats as described in (127). To show that $(E', \mathscr{H}')$ is a matroid, the principal facts that need to be checked are:
(i) that every two distinct hyperplanes intersect in a $j$-flat for some $j\leq 3$; and (ii) that for $0\leq j\leq 3$ every $j$-flat $F$ and element $x\in E-F$ lie in a common $(j+1)$-flat. The verification of (i) and (ii) is a straightforward matter of consideration by cases, and is left to the reader.

The preceding examples suggest the intricacy and variety of structure possible in matroid designs. Yet it is a remarkable fact that for large classes of integers $\gamma$, no matroid designs (other than trivioids) exist with cocircuit cardinality $\gamma$. And for many other classes of integers, all matroid designs with cocircuit cardinalities of that class can be described completely.

These and other results in the theory of matroid designs may be found in [14].

## 5. Appendix. Nontrivial $D$-Consistent Sequences With Leading Term $N$, $1\leq N\leq 40$, and Associated PMD's

We list below all nontrivial $D$-consistent sequences of maximal length with leading term $N$, $1\leq N\leq 40$. Those values of $N$, $1\leq N\leq 40$, for which no such sequences exist are not listed. Each nontrivial $D$-consistent sequence $D'$ having leading term $N$ is an initial subsequence of some maximal such sequence $D$. The known PMD's having $d$-sequence $D'$ are then listed after $D$ and the symbol $(m)$, where $m$ is the length of $D'$. ($m\geq 3$ since $D'$ is nontrivial.)

A $t-(v, k, \lambda)$ design (see Sec. 3.2) is denoted here by $D_t(v, k, \lambda)$. $PG(n, s)$ and $EG(n, s)$ denote, respectively, the finite projective and affine geometries of dimension $n$ and order $s$, and $IP(s)$ denotes an inversive plane of order $s$. The inversive planes are discussed in (67). Finally, $M^{(l)}$ denotes the truncation of the matroid $M$ at level $l$ (see (37)).

Sources for the construction of all PMD's listed are given by paragraph number or reference number. In several cases it may be proved that some initial subsequence of a $D$-consistent sequence $D$ is not realizable. This implies, by (81), that $D$ itself is not realizable. Reference is made in these cases to the appropriate theorems in the text.

| $N$ | Maximal $D$-consistent sequences | Initial subsequence lengths | Associated PMD's | References |
|---|---|---|---|---|
| 4 | 4, 2, 1, 1 | (3) | $PG(2, 2)$ | (72) |
| | | (4) | $EG(3, 2)$ | (77) |
| 6 | 6, 2, 1, 1, 1, 1 | (3) | $EG(2, 3)$ | (77) |
| | | (4) | $IP(3)$ | (67) |
| | | (5) | $D_4(11, 5, 1)$ | [13] |
| | | (6) | $D_5(12, 6, 1)$ | [13] |
| 8 | 8, 4, 2, 1, 1 | (4) | $PG(3, 2)$ | (72) |
| | | (5) | $EG(4, 2)$ | (77) |
| 9 | 9, 3, 1 | (3) | $PG(2, 3)$ | (72) |
| 10 | 10, 2, 1, . . ., 1 (8 terms) | (3) | $D_2(13, 3, 1)$ | [3] |
| | | (4) | $D_3(14, 4, 1)$ | [4] |
| | | (5)–(10) | unknown | |
| 12 | 12, 2, 1, . . ., 1 (10 terms) | (3) | $D_2(15, 3, 1)$ | [3] |
| | | (4) | $D_3(16, 4, 1)$ | [4] |
| | | (5)–(12) | unknown | |
| | 12, 3, 1, 1, 1 | (3) | $EG(2, 4)$ | (77) |
| | | (4) | $IP(4)$ | (67) |
| | | (5) | unknown | |
| 16 | 16, 2, 1, . . ., 1 (14 terms) | (3) | $D_2(19, 3, 1)$ | [3] |
| | | (4) | $D_3(20, 4, 1)$ | [4] |
| | | (5)–(16) | unknown | |
| | 16, 4, 1, 1, 1, 1 | (3) | $PG(2, 4)$ | (72) |
| | | (4) | $D_3(22, 6, 1)$ | [13] |
| | | (5) | $D_4(23, 7, 1)$ | [13] |
| | | (6) | $D_5(24, 8, 1)$ | [13] |
| | 16, 8, 4, 2, 1, 1 | (5) | $PG(4, 2)$ | (72) |
| | | (6) | $EG(5, 2)$ | (77) |
| 18 | 18, 2, 1, . . ., 1 (16 terms) | (3) | $D_2(21, 3, 1)$ | [3] |
| | | (4) | $D_3(22, 4, 1)$ | [4] |
| | | (5)–(18) | unknown | |
| | 18, 6, 2, 1 | (4) | $EG(3, 3)$ | (77) |
| 20 | 20, 4, 1, 1 | (3) | $EG(2, 5)$ | (77) |
| | | (4) | $IP(5)$ | (67) |
| 21 | 21, 3, 1, . . ., 1 (7 terms) | (3) | $D_2(25, 4, 1)$ | |
| | | (4)–(9) | unknown | |
| | 22, 2, 1, . . ., 1 (20 terms) | (3) | $D_2(25, 3, 1)$ | [3] |
| | | (4) | $D_3(26, 4, 1)$ | [4] |
| | | (5)–(22) | unknown | |
| 24 | 24, 2, 1, 1 | (3) | $D_2(27, 3, 1)$ | [3] |
| | | (4) | $D_3(28, 4, 1)$ | [4] |

| $N$ | Maximal $D$-consistent sequences | Initial subsequence lengths | Associated PMD's | References |
|---|---|---|---|---|
| | 24, 3, 1 | (3) | $D_2(28, 4, 1)$ | [5] |
| | 24, 4, 2, 1, 1 | (4) | $PG(4, 2)^{(4)}$ | (72) |
| | | (5) | $EG(5, 2)^{(5)}$ | (77) |
| 25 | 25, 5, 1 | (3) | $PG(2, 5)$ | (72) |
| 27 | 27, 9, 3, 1 | (4) | $PG(3, 3)$ | (72) |
| 28 | 28, 2, 1, . . ., 1 (26 terms) | (3) | $D_2(31, 3, 1)$ | [3] |
| | | (4) | $D_3(32, 4, 1)$ | [4] |
| | | (5) – (28) | unknown | |
| 30 | 30, 2, 1, . . ., 1 (28 terms) | (3) | $D_2(33, 3, 1)$ | [3] |
| | | (4) | $D_3(34, 4, 1)$ | [4] |
| | | (5) – (30) | unknown | |
| | 30, 5, 1, 1 | (3) | $EG(2, 6)$– does not exist | [3] |
| 32 | 32, 16, 8, 4, 2, 1 | (6) | $PG(5, 2)$ | (72) |
| | | (7) | $EG(6, 2)$ | (77) |
| 33 | 33, 3, 1 | (3) | $D_2(37, 4, 1)$ | [5] |
| 34 | 34, 2, 1, 1 | (3) | $D_2(37, 3, 1)$ | [3] |
| | | (4) | $D_3(38, 4, 1)$ | [4] |
| 36 | 36, 2, 1, . . ., 1 (34 terms) | (3) | $D_2(39, 3, 1)$ | [3] |
| | | (4) | $D_3(40, 4, 1)$ | [4] |
| | | (5) – (36) | unknown | |
| | 36, 3, 1, . . ., 1 (15 terms) | (3) | $D_2(40, 4, 1)$ | [3] |
| | | (14) – (17) | unknown | |
| | 36, 4, 1, . . ., 1 (8 terms) | (3) | $D_2(41, 5, 1)$ | [3] |
| | | (4) – (10) | unknown | |
| | 36, 6, 1 | (3) | $PG(2, 6)$– does not exist | [3] |
| | 36, 4, 2, 1, . . ., 1 (6 terms) | (4) – (9) | does not exist | (75) |
| 40 | 40, 2, 1, . . ., 1 (38 terms) | (3) | $D_2(43, 3, 1)$ | [3] |
| | | (4) | $D_3(44, 4, 1)$ | [4] |
| | | (5) – (40) | unknown | |
| | 40, 4, 1, 1 | (3) | $D_2(45, 5, 1)$ | [5] |
| | | (4) | unknown | |
| | 40, 5, 1 | (3) | unknown | |

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(Paper 77–376)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 1 & 2, January–June 1973

# Extrapolation Techniques Related to Transcendence Proofs*

**Kenneth B. Stolarsky****

(November 16, 1972)

An extrapolation method used by A. Baker to study linear forms in the logarithms of algebraic numbers is further refined, and used to study a general extrapolation problem involving a function holomorphic in a large disk.

Key words: Baker's extrapolation method; extrapolation; Hermite interpolation formula; transcendental numbers; transcendence proofs.

## 1. Introduction

Say that $f \in \mathfrak{F}$, a set of very smooth functions which are "discrete" (e.g., assume integral values at the integers) and "flat" (e.g., $f^{(i)}(0) = 0$, $0 \leq i \leq I$ and $|f'(x)|$ grows slowly). Then it is possible that $f(j) = 0$ for $0 \leq j \leq \phi(I)$, $j$ an integer, where $\phi(x) \to \infty$ as $x \to \infty$. For example, say $f \in \mathfrak{F}$ implies $f(j)$ is integral for integral $j$, $f(0) = 0$, and $|f'(x)| < 1$. Then $f(j) = 0$ for every integral $j$. Many proofs that specific numbers are transcendental involve showing that a certain discrete flat entire function of exponential type has a large number of consecutive integral zeros ([2], pp. 102–106; [1],[1] pp. 211–215; note the equation

$$\Psi(l) = 0, \qquad 1 \leq l \leq (L+1)^n$$

on p. 215). This paper shows a general such result.

References in this paragraph are to [1]; however our notation is different. As in [1] our proof uses an iterated extrapolation procedure based on a Hermite type formula like (15), p. 212. As in Lemma 4, p. 211, the general step in the procedure is to show that if $f^{(i)}(j)$ is small for $0 \leq i \leq I$, $0 \leq j \leq J$ then it is also small for $0 \leq i \leq I'$, $0 \leq j \leq J'$ where $I' < I$ but $J < J'$. In Baker's paper $I' = I/2$ so $I$ decreases *exponentially* with $u$, the number of iterations. In our paper $I$ decreases only *linearly* with $u$, so the extrapolation can be carried out much further. For a similar technique see [3].

Our theorem is stated in section 2; (ii) and (iii) correspond to flatness and discreteness respectively. The proof is given in section 3. In section 4 our extrapolation procedure is explained independently of the theorem, and shown to be optimal in some sense.

## 2. The Theorem

Let $\Psi = \langle A, I, J, \sigma, \tau; g, r \rangle$ be a set of positive real numbers with $A, I, J, \sigma, \tau \geq 1$ and $1 + I^{-1}\sigma \leq r$. Let $i$ and $j$ denote integers.

AMS Subject Classification: Primary 10 F 35, Secondary 30 A 80.
*An invited paper. Partially supported by NSF Contract GP 12363.
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[1] Figures in brackets indicate the literature references at the end of this paper.

DEFINITION: Let $\mathfrak{F}=\mathfrak{F}(\Psi)$ be the set of all functions $\varphi(z)$ holomorphic in $|z|\leq r$ such that

$$\text{(i)}\ |\varphi^{(i)}(z)|\leq \exp\{A\tau(|z|+1)\} \qquad \text{for } 0\leq i\leq I,$$
$$\text{(ii)}\ |\varphi^{(i)}(j)|\leq e^{-\sigma} \text{ for } 0\leq i\leq I, 0\leq j\leq J-1,$$

and

$$\text{(iii) if } |\varphi^{(i)}(j)|\leq e^{-\tau} \text{ then } |\varphi^{(i)}(j)|\leq e^{-\sigma} \text{ provided } 0\leq i\leq I \text{ and } 0\leq j\leq g.$$

THEOREM: *If* $\varphi\epsilon\mathfrak{F}$ *then* $|\varphi(\mathrm{j})|\leq \mathrm{e}^{-\sigma}$ *for*

$$0\leq \mathrm{j}\leq \mathrm{J}\ min\left\{\frac{1}{5}exp\ (\mathrm{I}/2000\mathrm{A}\tau),\ \frac{1}{5}(\sigma\tau/10\mathrm{JI}^2)^{1/3},\ \mathrm{g/J},\ \mathrm{r/J}\right\}. \tag{2.1}$$

The theorem is trivial unless

$$I>2000\,A\tau \tag{2.2}$$

and

$$\sigma>10\tau, \tag{2.3}$$

so we may assume (2.2) and (2.3) in our proof. The function

$$\varphi(z)=\exp\,(-\sigma-1+J^{-1}z)$$

shows that $\mathfrak{F}(\Psi)$ is nonempty when

$$\Psi=\{A, I, J, \sigma, \tau; J, \infty\}.$$

Two special cases of interest are

$$\Psi=\{1, \tau^2, 1, e^{\tau}, \tau; e^{\tau}, \infty\} \tag{2.4}$$

and

$$\Psi=\{1, \tau^{1+\epsilon}, 1, \tau^{1+3\epsilon}, \tau; \tau^{\epsilon}, \infty\} \tag{2.5}$$

where $\epsilon>0$. For (2.4) the inequality (ii) extends to very large values of $j$ when $i=0$, while for (2.5) a nontrivial conclusion is based on a very weak discreteness hypothesis (iii). It is clear, of course, that we need hypothesis (iii); if it were dropped then

$$\varphi(z)=\exp\{-e^{\tau}+\tau(\tau+1)^{-1}z\}$$

would satisfy the hypothesis but not the conclusion of the theorem in the case (2.4).

By setting $J=\tau=1$ and $\sigma=g=r=\infty$ in the theorem, we obtain the following

COROLLARY: *Let* A, $\mathrm{I}\geq 1$ *and let* i *and* j *denote integers. Say* $\varphi(z)$ *is an entire function such that for* $0\leq \mathrm{i}\leq \mathrm{I}$,

$$\text{(i)}\ |\varphi^{(\mathrm{i})}(\mathrm{z})|\leq exp\,\{\mathrm{A}(|\mathrm{z}|+1)\}$$

$$\text{(ii)}\ \varphi^{(\mathrm{i})}(0)=0$$

and

$$\text{(iii) } \textit{if } |\varphi^{(l)}(j)| < \frac{1}{4} \textit{ then } \varphi^{(l)}(j) = 0.$$

*Then* $\varphi(j) = 0$ for $0 \leq j \leq \frac{1}{5} exp$ (I/2000A).

A peculiarity of our theorem is that "beyond some point" knowing that $I$ is extremely large does not strengthen the conclusion. In applications it might be advisable to replace a given $I$ by a smaller one (however, see the comment regarding $\lambda$ at the beginning of sec. 4).

## 3. The Proof

Let $[x]$ denote the greatest integer in $x$, $I_1 = [I]$, $J_1 = [J]$, and let $i, j, s, t, u$ denote integers. The first two lemmas below are stated for the sake of completeness.

LEMMA 1: *Say* $|\zeta - j| = \frac{1}{2}$, $0 \leq j \leq J-1$, *and* $0 \leq z \leq S$ *where* j, J, S *are integers and* $\zeta$ *is a complex number. Then*

$$|\Pi| \leq 8 \cdot 2^{S+4J}$$

*where*

$$\Pi = \prod_{\nu=0}^{J-1} \left( \frac{z-\nu}{\zeta-\nu} \right). \tag{3.1}$$

LEMMA 2: (The Hermite interpolation formula). *Let the simple closed contour* C *contain* z *and* $0, 1, \ldots, J-1$. *If* $\varphi(z)$ *is analytic on and inside* C, *and* z *lies outside closed disks of radius* R *about* $0, 1, \ldots, J-1$, *then*

$$\varphi^{(l)}(z) = \sum_{j=0}^{J-1} \sum_{l=0}^{l'-1} \frac{\varphi^{(l+l)}(j)}{l!} \cdot \frac{1}{2\pi i} \int_{|\zeta-j|=R} (\zeta-j)^l (z-\zeta)^{-1} \tag{3.2}$$

$$\cdot \prod_{\nu=0}^{J-1} \left( \frac{z-\nu}{\zeta-\nu} \right)^{l'} d\zeta + \frac{1}{2\pi i} \int_C \varphi^{(l)}(\zeta) (\zeta-z)^{-1} \prod_{\nu=0}^{J-1} \left( \frac{z-\nu}{\zeta-\nu} \right)^{l'} d\zeta$$

$$= T_1 + T_2.$$

LEMMA 3: *Let* $\varphi \in \mathfrak{F}(\Psi)$. *Assume* (2.2), (2.3), *and part* (i) *of the definition. Assume also that*

$$|\varphi^{(i)}(j)| \leq e^{-\sigma} \textit{ for } 0 \leq i \leq I_u,\ 0 \leq j \leq J_u - 1 \tag{3.3}$$

where

1. $0 \leq I_u \leq I_1$, *and*
2. $1 \leq J_1 \leq J_u \leq \sigma\tau / 10 I_1^2$.

Then

$$\left| \varphi^{(0)}(j) \right| \leq e^{-2\tau} \tag{3.4}$$

provided

$$0 \leq t \leq [I_u - 650A\tau] \text{ and } 0 \leq j \leq [2.5J_u] + 1.$$

PROOF: We first show (3.4) for

$$0 \leq t \leq [\tfrac{1}{2}(I_u + s)] + 1,\ J_u \leq j \leq S_u = S_u(s) \tag{3.5}$$

where

$$0 \leq s \leq I_u - 1250A\tau \text{ and } S_u = [\tfrac{1}{2}(I_u - s)J_u/100A\tau]. \tag{3.6}$$

Then (3.4) follows by letting $s = s_0 = [I_u - 1250A\tau]$. Note that

$$S_u(s) \geq S_u(s_0) \geq 2.5J_u > 2J_u.$$

For the rest of the proof set $I' = [\frac{1}{2}(I_u - s)]$, $J = J_u$, $S = S_u$ and $\rho = \rho_u = 2S_u$. For $J \leq z \leq S$ and $0 \leq t \leq I'$ Lemma 2 is applicable for $C$ a circle of radius $\rho$ because

$$\rho = 2S \leq I_1 J/100A\tau \leq I_1 \sigma\tau/100A\tau \cdot 10I_1^2 \leq r$$

by the definition of section 2. Thus (3.2) holds and clearly $i + t \leq I_u$ in (3.2). Hence by Lemma 1, (3.3), and $S > 2J$ (set $R = \frac{1}{2}$),

$$|T_1| \leq I'Je^{-\sigma} \cdot \tfrac{1}{2} \cdot 2 \cdot 8^{I'}2^{I'(S+4J)} \leq e^{-\sigma}2^{10I'J + I'S}$$

$$\leq e^{-\sigma}2^{6I'S} < e^{-(1/2)\sigma}$$

since

$$6I'S \leq 6I_1 \cdot (I_1/200A\tau) \cdot \sigma\tau/10I_1^2 < \tfrac{1}{2}\sigma \cdot$$

Next,

$$|T_2| \leq \frac{\rho}{\rho - S}\ e^{A\tau(\rho+1)} \left(\frac{S}{\rho - J}\right)^{I'J}$$

$$\leq 2 \exp\left(A\tau + 2A\tau S - I'J \ln \tfrac{3}{2}\right).$$

Since $I' \geq 600A\tau > 1$, we have

$$2A\tau S \leq 2A\tau \cdot 2I'J/100A\tau = I'J/25.$$

Therefore ($J \geq 1$)

$$|T_2| \leq \exp(2A\tau - .36I') \leq \exp(2A\tau - 216A\tau) < e^{-200\tau}.$$

Finally, by (2.3),

$$|\varphi^{(t)}(z)| \leq |T_1| + |T_2| \leq e^{-5\tau} + e^{-200\tau} \leq e^{-2\tau}.$$

In particular this holds when $z = j$, so Lemma 3 is proved.

For the proof of the theorem, set

$$I_{u+1}=[I_u-650A\tau] \text{ and } J_{u+1}=[2.5J_u]+1$$

for $u \geq 1$. Since $e^{-2\tau} < e^{-\tau}$, whenever the conclusion of Lemma 3 is valid we expect by (iii) that

(3.7) $$|\varphi^{(i)}(j)| \leq e^{-\sigma} \text{ for } 0 \leq i \leq I_{u+1}, 0 \leq j \leq J_{u+1}-1.$$

Hypothesis (ii) asserts that (3.7) is valid for $u=0$. Thus (3.7) is valid for $0 \leq u \leq U$ by induction provided conditions 1. and 2. of Lemma 3 hold for $u \leq U$, and $J_{U+1} \leq \max(g, r)$ so (i) and (iii) are applicable. We choose $U=[\min(U_1, U_2)]$ where

$$U_1=I/1400A\tau \text{ and } U_2=\frac{1}{2}\ln(\sigma\tau/10JI^2).$$

The inequality $I_{u+1} \geq I_u-700A\tau$ implies $I_U \geq I_1-\frac{1}{2}I \geq 0$ so condition 1. follows. For 2., note that

$$2.5 \leq J_{u+1}/J_u \leq 3.5$$

so

$$(2.5)^U \leq J_{U+1}/J_1 \leq (3.5)^U \leq (\sigma\tau/10JI^2)^{(1/2)\ln(3.5)}.$$

Thus

$$J_{U+1} \leq J_1(\sigma\tau/10JI_1^2)$$

and condition 2. follows. If $U=[U_1]$ then

$$J_{U+1} \geq \frac{1}{2} J \exp\{((1400A\tau)^{-1}I-1)\ln 2.5\}$$

$$\geq \frac{1}{5} J \exp(I/2000A\tau).$$

If $U=[U_2]$ then

$$J_{U+1} \geq \frac{1}{2} J \exp\left\{\left(\frac{1}{2}\ln(\sigma\tau/10JI^2)-1\right)\ln 2.5\right\}$$

$$\geq \frac{1}{5} J(\sigma\tau/10JI^2)^{1/3}.$$

This proves the theorem.

## 4. The General Extrapolation Procedure

Proposition 4.2 (with $\lambda(x)=x$) idealizes our extrapolation procedure, and shows that the first term inside the min(,,,) of (2.1) cannot be improved in its general from (at least by these methods). Under certain conditions the second term inside the min(,,,) can be replaced by

$$(\sigma\lambda(\tau)/10I\lambda(I)J)^{1/3}$$

but we shall omit the details. If we set $t=0$ in (4.4) we get the old extrapolation procedure.

Let $\lambda(x)$ be an increasing positive function such that $\lambda'(1) > 0$ and $\lambda''(x) \leq 0$ (for example $\lambda(x) = x^{1/3}$). Let $\mu$ be the inverse function so that $\mu(\lambda(x)) = \lambda(\mu(x)) = x$.

PROPOSITION 4.1: *Let* $x \geq A > 0$ *and define*

$$\theta(x, A) = \min \sum_{n=1}^{N-1} \mu(A_{n+1}/A_n) \tag{4.1}$$

*where the minimum is over all integers* $N \geq 2$ *and all real* $N$*-tuples* $(A_1, \ldots, A_N)$ *such that* $A = A_1 \leq A_2 \leq \ldots \leq A_{N-1} \leq A_N = x$. *Then*

$$C_{20}(\mu) \ln (x/A) \leq \theta(x, A) \leq C_{21}(\mu) \ln (x/A) \tag{4.2}$$

*where* $C_{20}, C_{21}$ *depend only on* $\mu$.

PROOF: Let $A_n = x^{n/N} A^{(N-n)/N}$. Then $A_{n+1}/A_n = (x/A)^{1/N}$,

so

$$\theta(x, A) \leq \min_N \{(N-1)\mu(\{x/A\}^{1/N})\}.$$

The upper bound follows by taking $N = [\ln (x/A)]$ for $x$ large. On the other hand, by the convexity of $\mu$, the inequality of the arithmetic-geometric means, and some simple calculus,

$$\theta(x, A) \geq \min_N \left\{ (N-1)\mu\left(\frac{1}{N-1} \sum_{n=1}^{N-1} A_{n+1}/A_n\right)\right\}$$

$$\geq \min_N \{(N-1)\mu(\{x/A\}^{1/(N-1)})\}$$

$$\geq \min_N \{C_{22}(\mu)(N-1)(x/A)^{1/(N-1)}\} \geq C_{20}(\mu) \ln (x/A)$$

and the result follows.

PROPOSITION 4.2: *Let* S *be the smallest subset of* $[0, \infty] \times [0, \infty]$ *such that*

$$[0, J] \times [0, I] \subseteq S \text{ for some } I, J > 0. \tag{4.3}$$

(4.4) *If* $[0, A] \times [t, B] \subseteq S$, *then*

$$[0, A\lambda(B-t)] \times [t, \tfrac{1}{2}(B+t)] \subseteq S.$$

*Then*

$$Je^{C_{23}(\lambda)I} \leq \max \{x | (x, y) \in S\} \leq Je^{C_{24}(\lambda)I}$$

*where* $C_{23}(\lambda)$, $C_{24}(\lambda)$ *depend only on* $\lambda$.

PROOF: Let $x = J\lambda(I-t)$, $y = \frac{1}{2}(I+t)$. Then it is clear that $S = U_{i=0}^{\infty} S_i$ where $S_0 = [0, J] \times [0, I]$, $S_1$ consists of all points in the first quadrant under the curve

$$y = I - \mu(x/J),$$

$S_2$ of all such points under all curves

$$y = I - \tfrac{1}{2}\{\mu(A_2/J) + \mu(x/A_2)\} \qquad J \leq A_2 \leq x$$

and in general $S_N$ of all such points under

$$y = I - \tfrac{1}{2}\sum_{n=1}^{N-1} \mu(A_{n+1}/A_n)$$

where $A_1 = J$, $A_N = x$ and $A_1 \leq A_2 \leq \ldots \leq A_N$. In the notation of the previous lemma $S$ contains all points under the curve

$$y = I - \tfrac{1}{2}\,\theta(x, J).$$

This crosses the $x$-axis at $\theta^{-1}(2I)$, so the result follows from Proposition 4.1.

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(Paper 77B1 & 2–377)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 1 & 2, January–June 1973

# A Catalog of Minimal Blocks*

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(November 16, 1972)

In this paper we provide a catalog of the minimal blocks with 10 and fewer vertices, together with a discussion of the methods and theorems used to produce the catalog. In addition, we prove a theorem which is a strengthening of a similar theorem of Fleischner [2] on the structure of minimal blocks.

Key words: Blocks; combinatorics; examples of graphs; graph theory; minimal blocks; planar graphs; thickness of graphs; two-connected graphs.

## 1. Description of the Catalog

The majority of the definitions used here will be found in [3],[1] with the terms "point", "line", and "cycle" replaced by *vertex*, *edge*, and *circuit*. In particular, a *block* is a connected graph with no cut vertex, and a block is *minimal* if no spanning subgraph of the block with fewer edges is also a block. We consider the graph with one vertex and no edges (the *vertex graph*) and the graph with two vertices and a single edge joining them (the *link graph*) to be minimal blocks. To distinguish between a path $p$ and the graph which contains exactly the edges and vertices of $p$, we denote the graph by $|p|$; for simplicity of terminology, we will refer to both $p$ and $|p|$ as "paths." If $p$ is a path, $F(p)$ denotes the first vertex of $p$, and $L(p)$ denotes the last vertex of $p$. The undirected edge joining vertices $\rho$ and $\sigma$ is denoted by $(\rho, \sigma)$ or $(\sigma, \rho)$ interchangeably.

Let $A_1, \ldots, A_k$ be $k$ disjoint graphs which are paths, with $k \geq 2$, such that path $A_i$ contains $m_i$ vertices, for each $i \in \{1, \ldots, k\}$, with $m_1 \geq m_2 \geq \ldots \geq m_k \geq 1$. Let $\alpha$ and $\beta$ be vertices not in $\bigcup_{i=1}^{k} V(A_i)$. For each $i \in \{1, \ldots, k\}$, let $a_i$ be a Hamiltonian path in $A_i$. Let the graph $P(m_1, m_2, \ldots, m_k)$ be the graph with $V(P) = \bigcup_{i=1}^{k} V(A_i) \cup \{\alpha, \beta\}$ and $X(P) = \bigcup_{i=1}^{k} [X(A_i) \cup \{(\alpha, F(a_i)), (L(a_i), \beta)\}]$ (where $X(G)$ denotes the edge set of graph $G$) such that $A_1, \ldots, A_k$ are subgraphs of $P$. We call $P(m_1, \ldots, m_k)$ a *partition graph*; clearly a partition graph with $n$ vertices is completely determined up to isomorphism by a partition of the integer $n-2$. Further, it is clear that each partition graph is a minimal block.

In the sequence $m_1, \ldots, m_k$, if $m_{s+1} = m_{s+2} = \ldots = m_{s+r}$ for some integer $s$ and integer $r \geq 2$, we may write $m_1, \ldots, m_k$ in the form $m_1, \ldots m_s, r \times m_{s+1}, m_{s+r+1}, \ldots, m_k$. For example, $P(3, 4 \times 2)$ is shown in figure 1. Using this notation, the catalog at the end of this paper gives all minimal blocks with 10 and fewer vertices. Above each of the drawings of a graph with 7 or more vertices is a sequence in parentheses. This sequence is the degree sequence of the associated graph, i.e., the sequence of degrees of the vertices of the graph in descending order. The

*AMS Subject Classification:* 0540.
*An invited paper. Partial support for this research came through Professor W. T. Tutte from the National Research Council of Canada.
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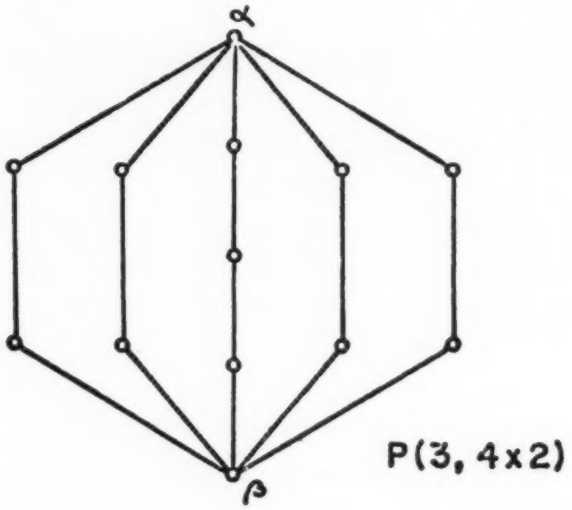
[1] Figures in brackets indicate the literature references at the end of this paper.

minimal blocks presented in the catalog are all planar simply because both of the smallest nonplanar minimal blocks ($S(K_5)$ and $S(K_{3,3})$) have 15 vertices.




## 2. Theory and Method for Constructing the Catalog

A *suspended path* in a graph $G$ is a path $p$ of $G$ such that all of the vertices in $V(p) - \{F(p), L(p)\}$ have degree 2 in $G$. The removal from and addition to a graph of a suspended path is similar to the removal from and addition to a graph of an edge. A graph $G$ with blocks $B_1, \ldots, B_k$ is a *block-path chain* iff *either* $k = 1$ *or* $k > 1$, $B_i$ and $B_j$ have exactly one common vertex if $j \in \{i-1, i+1\}$, and $B_i$ and $B_j$ have no common vertices if $j \notin \{i-1, i, i+1\}$. Following [1], two vertices $\alpha$ and $\beta$ in a graph $G$ are *compatible* in $G$ iff, for every path $p$ with $F(p) = \alpha$ and $L(p) = \beta$, two vertices of $p$ are joined by an edge of $G$ iff the edge is in $p$.

Whitney [6] proved that in any block with three or more vertices there is an edge or a suspended path whose removal from the block results in a smaller block or a link graph. Thus a block with $k$ vertices and $m$ edges can be constructed from one with $k-1$ or fewer vertices or $k$ vertices and $m-1$ edges by a process of adding an edge or a suspended path joining some distinct pair of vertices. If the block to be constructed is to be minimal, the removal of an edge from the constructed block cannot yield a block; thus each minimal block with $k$ vertices can be constructed from a block with $k-1$ or fewer vertices by the addition of a suspended path joining some pair of distinct vertices. Further, it is easily shown that if $G$ is a minimal block constructed from a block $B$ by the addition of a suspended path joining some pair of distinct vertices of $B$, then $B$ is a minimal block.

Dirac [1] has shown that removing an edge from a minimal block $G$ results in a block-path chain in which any two nonadjacent cut vertices are compatible. Further, he has shown that if $S$ is a block of this block-path chain, then either $S$ is a link graph or the cut vertices in $S$ of the block-path chain are not adjacent.[2] In particular, if a minimal block $G$ is constructed from a block $B$ by the addition of a suspended arc, then either $B$ is a link graph or the two vertices in $B$ of the suspended path are not adjacent in $G$ and are compatible in $B$.

[2] Dirac also showed that any circuit in a minimal block contains a vertex of degree two. This fact can be used to give a very short proof of the theorem that a nonplanar minimal block has thickness two, where the thickness of a graph $G$ is the smallest number of planar subgraphs of $G$ whose union is $G$. See also Plummer [4] and Tutte [5].

Thus to construct all minimal blocks with $k$ vertices, it is sufficient to find all minimal blocks with $k-1$ or fewer vertices, identify all nonadjacent compatible pairs of vertices in each of these minimal blocks, and join each such pair of vertices with a suspended path of length sufficient to produce a graph with $k$ vertices. The resulting collection of graphs can be reduced by eliminating all but one graph in each isomorphism class of graphs.

The program described in the preceding paragraph is the one used for finding the catalog included in this paper. Some reductions in the labor of carrying out the program are easily found. It is easy to show that if the removal of two vertices from a minimal block results in a disconnected graph, then the vertices are compatible. Most compatible pairs of vertices can be detected simply by finding all such sets of two vertices. Further, the partition graphs with $k$ vertices can be found immediately by finding all partitions of the integer $k-2$ into two or more parts. Since a partition graph can only be obtained from the link graph or from a smaller partition graph by the addition of a suspended path, it is easy to avoid producing any further copies of these particular graphs. Finally, it is clear that if there is an automorphism of a graph $B$ which carries one pair of vertices of $B$ into another pair of vertices of $B$, and if $G$ ($G'$) is formed from $B$ by joining the first (second) pair of vertices by a suspended path of length $m$, then $G$ and $G'$ are isomorphic. Thus many isomorphic copies of graphs can be avoided by using only one compatible pair of vertices from a block $B$ among all those pairs which can be carried into one another by automorphisms of $B$.

Once all minimal blocks with $k$ vertices have been found, the job of eliminating the isomorphic copies produced in spite of the earlier tricks is considerably eased by first classifying the graphs by their degree sequences. Within each class, the isomorphic copies can then be eliminated with a small amount of additional labor.

## 3. Structure of Minimal Blocks

An *end block* of a block-path chain $G$ is a block of $G$ which contains at most one cut vertex of $G$. Following the terminology of [2], given a graph $G$, $D(G)$ is the set of all edges of $G$ which are not incident with vertices whose degree in $G$ is two. A *DT-subgraph* of a graph $G$ is a subgraph $H$ of $G$ such that every edge of $H$ is incident with a vertex whose degree *in* $G$ is two. In [2], Fleischner proved a theorem which can be put in the following form:

THEOREM A: *Let* $\mathrm{G}$ *be a minimal block with* $|\mathrm{D(G)}| \geq 2$, *and suppose* $\{\lambda_0, \lambda_1\} \subseteq \mathrm{D(G)}$ *with* $\lambda_1$ *contained in an end block* $\mathrm{B}_0$ *of* $\mathrm{G}-\lambda_0$. *Then an end block of* $\mathrm{G}-\lambda_1$ *is a subgraph of* $\mathrm{B}_0$.

Using this result, Fleischner obtained an interesting theorem on the structure of minimal blocks. The following theorem is a strengthening of his theorem, using a modification of his proof.

THEOREM: *Let* $\mathrm{G}$ *be a finite minimal block with* $|\mathrm{D(G)}| \geq 1$. Then either

(1) *there exists* $\lambda \in \mathrm{D(G)}$ *such that both end blocks of* $\mathrm{G}-\lambda$ *are* DT*-subgraphs of* G, *or*

(2) *there exists* $\{\lambda, \mu\} \subseteq \mathrm{D(G)}$ *such that one end block* $\mathrm{B}_\lambda$ *of* $\mathrm{G}-\lambda$ *and one end block* $\mathrm{B}_\mu$ *of* $\mathrm{G}-\mu$ *are* DT*-subgraphs of* G *and* $\mathrm{V(B}_\lambda) \cap \mathrm{V(B}_\mu)$ *contains at most one vertex.*

PROOF: If $|D(G)|=1$, the theorem is trivial. Suppose $|D(G)| \geq 2$ and suppose that (1) does not hold, so that for every $\lambda \in D(G)$, one of the end blocks of $G-\lambda$ is not a *DT*-subgraph of $G$.

Let $\lambda_0$ be an element of $D(G)$ with end blocks $B_{1,0}$ and $B_{2,0}$ in $G-\lambda_0$ such that $B_{2,0}$ is not a *DT*-subgraph of $G$. For $i \in \{1, 2\}$, if $B_{i,0}$ is not a *DT*-subgraph of $G$, let $\lambda_{i,1} \in X(B_{i,0}) \cap D(G)$ (recall that $X(B)$ is the set of edges of graph $B$). By Theorem A, one end block $B_{i,1}$ of $G-\lambda_{i,1}$ is a subgraph of $B_{i,0}$. Continuing, for $i \in \{1, 2\}$, if $B_{i,j}$ is not a *DT*-subgraph of $G$, then $X(B_{i,j}) \cap D(G)$ is not empty; let $\lambda_{i,j+1} \in X(B_{i,j}) \cap D(G)$. Then by Theorem A, one end block $B_{i,j+1}$ of $G-\lambda_{i,j+1}$ is a subgraph of $B_{i,j}$. Since $G$ is finite, we must eventually reach an edge $\lambda_{i,k} \in X(B_{i,k-1}) \cap D(G)$ such that one end block $B_{i,k}$ of $G-\lambda_{i,k}$ is a subgraph of $B_{i,k-1}$ and is a *DT*-subgraph of $G$. But since $B_{i,j}$ is a subgraph of $B_{i,j-1}$ for all $j \in \{1, \ldots, k\}$, $B_{i,k}$ is a subgraph of $B_{i,0}$.

If $B_{1,0}$ is a *DT*-subgraph of $G$, let $\lambda = \lambda_0$ and $B_\lambda = B_{1,0}$. If $B_{1,0}$ is not a *DT*-subgraph of $G$, by the above argument there exists $\lambda \in X(B_{1,0}) \cap D(G)$ such that an end block $B_\lambda$ of $G-\lambda$ is a subgraph of $B_{1,0}$ and $B_\lambda$ is a *DT*-subgraph of $G$. Also there exists $\mu \in X(B_{2,0}) \cap D(G)$ such that an end block $B_\mu$ of $G-\mu$ is a subgraph of $B_{2,0}$ and is a *DT*-subgraph of $G$. Then, since $V(B_{1,0}) \cap V(B_{2,0})$ contains at most one vertex, $V(B_\lambda) \cap V(B_\mu)$ also contains at most one vertex.

My thanks are due to Professor William T. Tutte and to Professor Crispin St. J. A. Nash-Williams for their help and support during the preparation of this paper.

## 4. Appendix. Catalog of Minimal Blocks With 10 and Fewer Vertices

### 1 VERTEX

### 2 VERTICES



### 3 VERTICES

### 4 VERTICES

### 5 VERTICES

P(2, 1)

P(1, 1, 1)

### 6 VERTICES

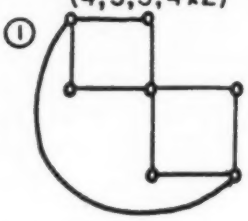
P(2, 2)

P(2, 1, 1)

P(1, 1, 1, 1)

### 7 VERTICES

(4,3,3,4x2)



P(3,2)

P(3,1,1)

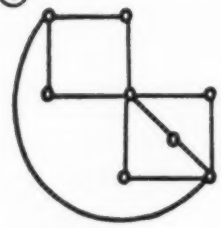
P(2,2,1)

P(2,1,1,1)

P(5x1)

### 8 VERTICES

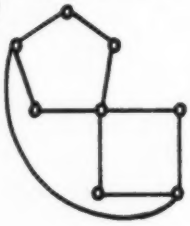
(5,4,3,5x2)



P(3,3)

P(4,1,1)

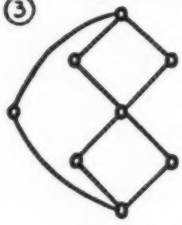
(4,3,3,5x2)



P(3,2,1)

P(3,1,1,1)

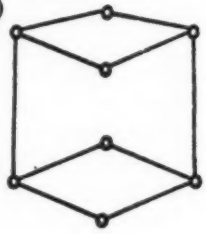
(4,3,3,5x2)



P(2,2,2)

P(2,2,1,1)

(4x3,4x2)

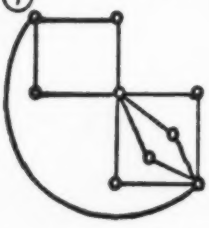


P(2,4x1)

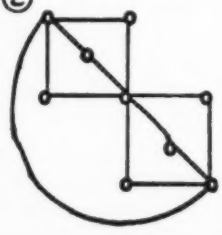
P(6x1)

### 9 VERTICES

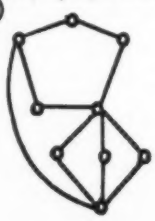
(6,5,3,6x2)



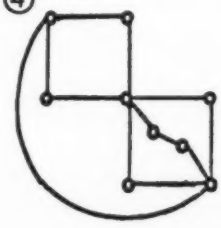
(6,4,4,6x2)



(5,4,3,6x2)



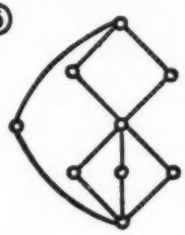
(5,4,3,6x2)



## 9 VERTICES (CONTINUED)

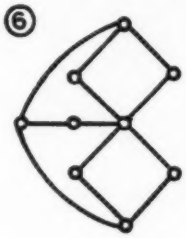
(5,4,3,6x2)

⑤



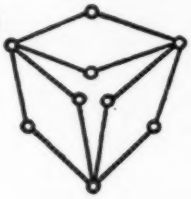
(5,3,3,3,5x2)

⑥



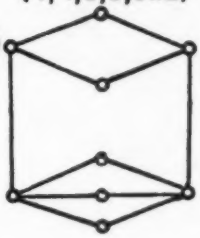
(4,4,4,6x2)

⑦



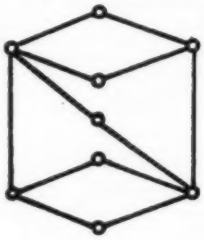
(4,4,3,3,5x2)

⑧



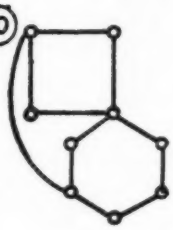
(4,4,3,3,5x2)

⑨



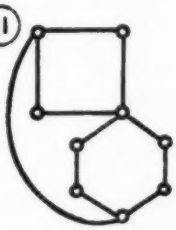
(4,3,3,6x2)

⑩



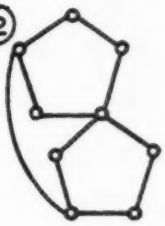
(4,3,3,6x2)

⑪



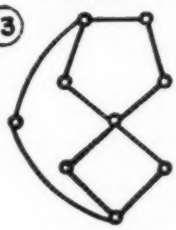
(4,3,3,6x2)

⑫



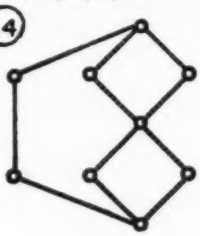
(4,3,3,6x2)

⑬



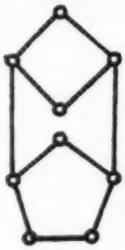
(4,3,3,6x2)

⑭



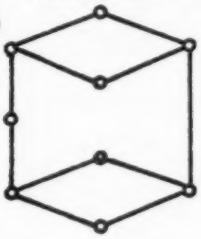
(4x3,5x2)

⑮



(4x3,5x2)

⑯



P(4,3)
P(5,1,1)
P(4,2,1)

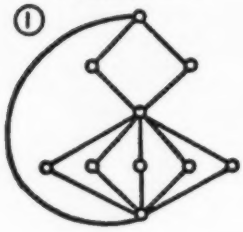
P(3,2,2)
P(3,3,1)
P(4,1,1,1)

P(3,2,1,1)
P(2,2,2,1)
P(3,4x1)

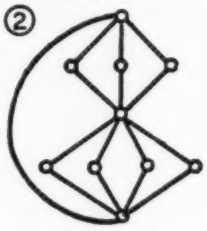
P(2,2,1,1,1)
P(2,5x1)
P(7x1)

## 10 VERTICES

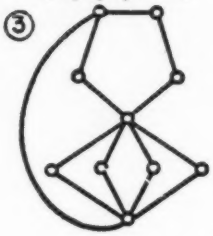
(7,6,3,7x2)



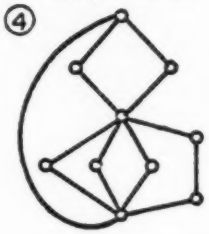

(7,5,4,7x2)



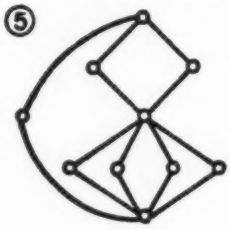

(6,5,3,7x2)




(6,5,3,7x2)

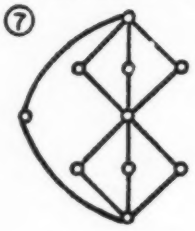



(6,5,3,7x2)

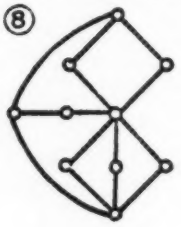



(6,4,4,7x2)

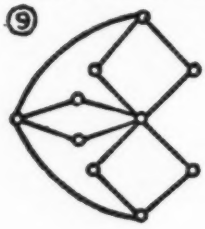
(6,4,4,7x2)



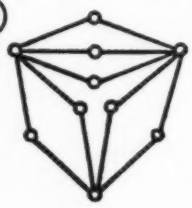

(6,4,3,3,6x2)



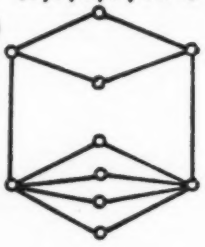

(6,4,3,3,6x2)



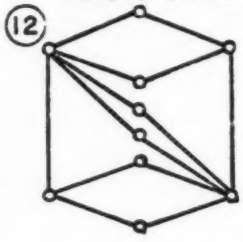

(5,5,4,7x2)



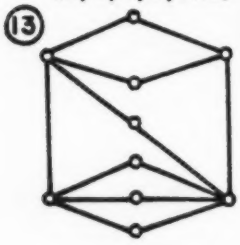

(5,5,3,3,6x2)



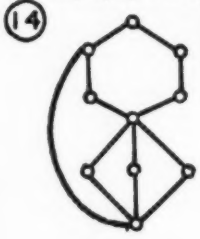

(5,5,3,3,6x2)



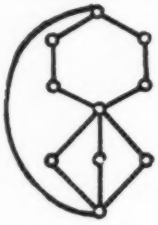

(5,4,4,3,6x2)




(5,4,3,7x2)




(5,4,3,7x2)

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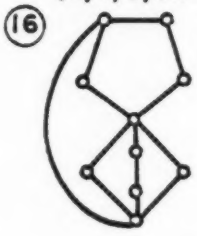
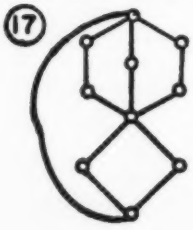
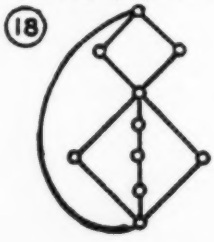


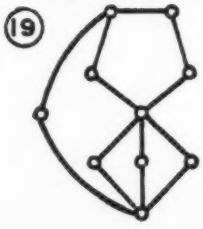

(5,4,3,7x2)



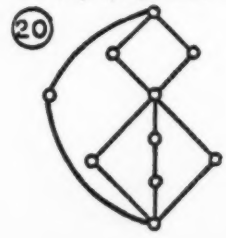

(5,4,3,7x2)




(5,4,3,7x2)

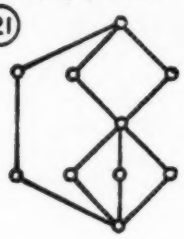



(5,4,3,7x2)



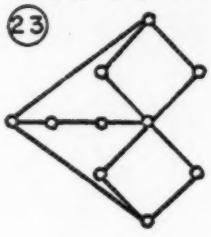
## 10 VERTICES (CONTINUED)

21 (5,4,3,7x2)

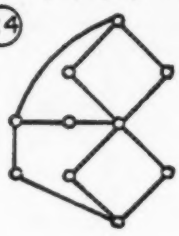


22 (5,3,3,3,6x2)

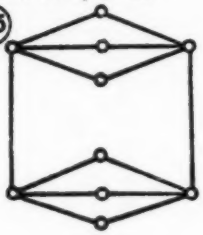
23 (5,3,3,3,6x2)



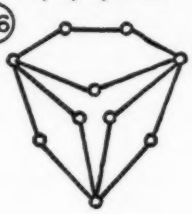
24 (5,3,3,3,6x2)



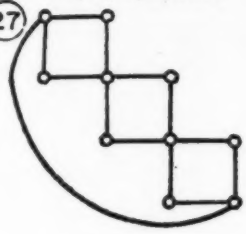
25 (4x4,6x2)



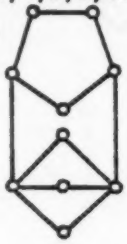
26 (4,4,4,7x2)



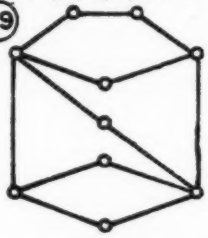
27 (4,4,3,3,6x2)



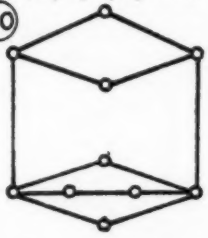
28 (4,4,3,3,6x2)



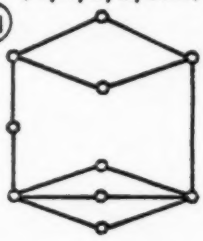
29 (4,4,3,3,6x2)



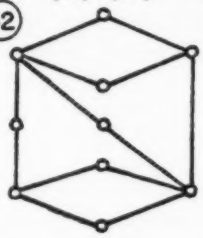
30 (4,4,3,3,6x2)



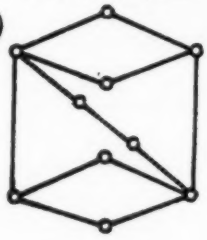
31 (4,4,3,3,6x2)



32 (4,4,3,3,6x2)



33 (4,4,3,3,6x2)



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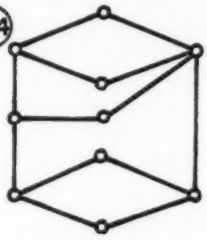
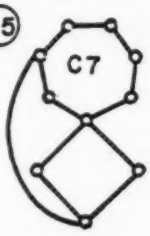


35 (4,3,3,7x2)

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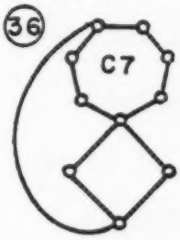
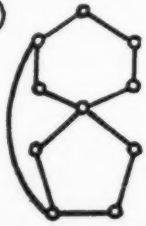
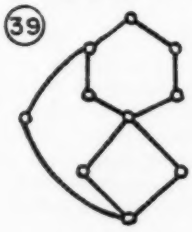



37 (4,3,3,7x2)



38 (4,3,3,7x2)

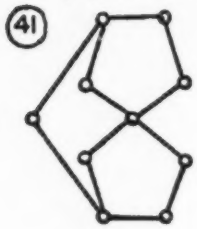
39 (4,3,3,7x2)



40 (4,3,3,7x2)

## 10 VERTICES (CONTINUED)

(4,3,3,7x2)

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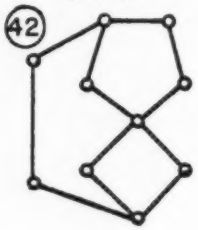


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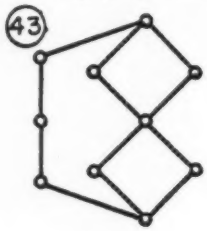
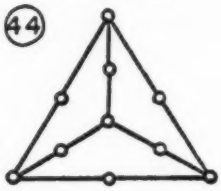



(4x3, 6x2)

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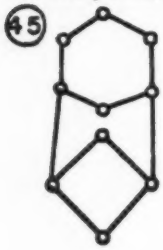


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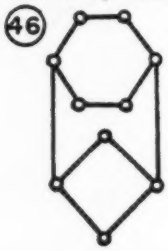


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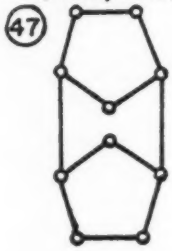


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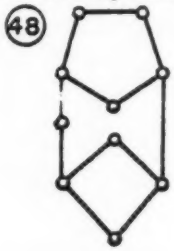


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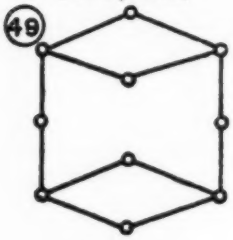


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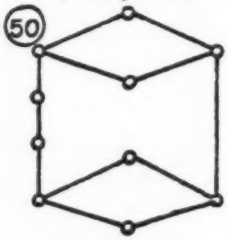


| | | | |
|---|---|---|---|
| P(4,4) | P(4,3,1) | P(3,3,1,1) | P(3,5x1) |
| P(6,1,1) | P(3,3,2) | P(4x2) | P(2,2,4x1) |
| P(5,2,1) | P(5,1,1,1) | P(4,4x1) | P(2,6x1) |
| P(4,2,2) | P(4,2,1,1) | P(3,2,1,1,1) | P(8x1) |
| | P(3,2,2,1) | P(2,2,2,1,1) | |

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(Paper 77B1 & 2–378)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 1 & 2, January–June 1973

# On the Eigenvalues of A + B and AB

**Helmut Wielandt** *

**(December 16, 1972)**

Using the usual field of values and the angular field of values inclusion sets are found for the eigenvalues of sums and products of $n \times n$ complex matrices. For instance if the field of values of $B$ does not contain 0 it is found that the quotient of the field of values of $A$ by that of $B$ contains the eigenvalues of $AB^{-1}$. Applications are made to the polar form ($AB$ where $A$ is unitary and $B$ positive semidefinite) and to products $AB$ with $A$ hermitian and $B + B^*$ positive definite.

Key words: Angular field of values; eigenvalues; field of values; hermitian; inertia; positive semidefinite.

This paper contains some simple results connecting the location of the complex eigenvalues of the sum and product of two $n \times n$ matrices $A, B$ with certain point sets assigned to $A$ and $B$: the familiar field of values and the angular field.

If $M$ is an $n \times n$ complex matrix, its field of values is $F(M) \equiv \{x^*Mx |\ x \text{ a complex } n\text{-vector}, x^*x = 1\}$.

$F(M)$ is known to be a closed bounded convex set containing all the eigenvalues of $M$; $F(M)$ coincides with the convex closure of the eigenvalues if $M$ is normal. Apart from $F(M)$ we introduce the angular field $W(M)$:

$$W(M) \equiv \{x^*Mx |\ x \neq 0 \text{ complex } n\text{-vector}\}.$$

$W(M)$ clearly contains $F(M)$ and is the union of $F(M)$ together with all open rays with origin 0 containing a point $\neq 0$ of $F(M)$. Hence $W(M)$ has one of the following shapes: (1) the whole complex plane; (2) an angular area (not necessarily containing 0) with vertex 0 and angle $\alpha$, $0 < \alpha \leqslant \pi$; (3) a half line starting from (and not necessarily containing) 0; (4) a line through 0; and (5) the single point 0. Cases (3) and (4) occur when $M = e^{i\theta}H$ and $H \neq 0$ is hermitian. It is easy to see that $W(N^*MN) = W(M)$ if $\det N \neq 0$, $W(M^*) = \overline{W(M)}$, and $W(M^{-1}) = \overline{W(M)}$ if $\det M \neq 0$.

In order to state our results easily, we use the following notation. If $S_1$ and $S_2$ are sets of complex numbers, we denote by $S_1 + S_2$, $S_1S_2$, $S_1/S_2$, respectively the set of all numbers of the form $s_1 + s_2$, $s_1s_2$, $s_1/s_2$ for $s_i \in S_i$ (in case of $S_1/S_2$ we assume $0 \notin S_2$). Clearly, we have $F(A+B) \subseteq F(A) + F(B)$ and $W(A + B) \subseteq W(A) + W(B)$ since $x^*\ (A + B)\ x = x^*Ax + x^*Bx$.

Let us start with an obvious theorem concerning the eigenvalues of $A+B$.

THEOREM 1: *If* $\lambda$ *is an eigenvalue of* $A+B$, *then* $\lambda \in F(A) + F(B)$.

*Proof*: $\lambda \in F(A+B) \subseteq F(A) + F(B)$.

We proceed to search for similar theorems holding for products of matrices. Unfortunately the corresponding inclusion $F(AB) \subseteq F(A)F(B)$ is not in general valid; a counterexample will appear in remark 3 to follow.

However, we may note that quotients of matrices behave nicely.

THEOREM 2: *Let* $0 \notin F(B)$. *If* $\lambda$ *is an eigenvalue of* $B^{-1}A$ *or of* $AB^{-1}$, *then* $\lambda \in F(A)/F(B)$.

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AMS Subject Classification: 1555.

* An invited paper. This is a minor revision of a paper that has been referred to many times but has been very difficult to obtain since it appeared as an NBS Report No. 1367 in 1951. A good deal of work on closely related topics has appeared since then. The reader may therefore be interested in a partial bibliography at the end of this paper that deals primarily with spectra of products and facts about inertia. The wide range of references given in each of these articles is also recommended. The bibliography was compiled by C. R. Johnson at NBS. Present address of the author: University of Tübingen, Germany.

PROOF: Since $0 \notin F(B)$, 0 is not an eigenvalue of $B$, so $B$ is nonsingular. Let $B^{-1}Ax = \lambda x$, $\|x\| = 1$. Then $Ax = \lambda Bx$, $x^*Ax = \lambda x^*Bx$, $x^*Bx \neq 0$; hence $\lambda \in F(A)/F(B)$. In the case of $AB^{-1}$ we may refer to the well-known theorem stating that $AB^{-1}$ and $B^{-1}A$ have the same eigenvalues, or we may repeat the proof given above using a left-hand eigensolution $x^*$ instead of $x$.

From theorem 2 we deduce the corresponding theorem for $AB$ under suitable additional conditions.

THEOREM 3: *Let* A *be arbitrary, and* B *hermitian and positive semidefinite. If* $\lambda$ *is an eigenvalue of* AB, *then* $\lambda \in F(A)F(B)$.

COROLLARY 3′: *Let* A *be unitary with eigenvalues* $\alpha_1, \ldots, \alpha_n$; *let* B *be hermitian and positive semidefinite with eigenvalues* $\beta_1 \leq \ldots \leq \beta_n$ *and let* $\lambda$ *be any eigenvalue of* AB *or* BA. *Then* $\beta_1 \leq |\lambda| \leq \beta_n$, *and if all the* $\alpha_i$ *are contained in an arc* $\Phi$ *of the unit circle of length* $\leqq \pi$, *then* $\arg \lambda \in \Phi$.

The statements on $\beta_n$ and $\arg \lambda$ follow immediately from theorem 3 since $|\alpha_i| = 1$ and $F(A)$ is the convex closure of the $\alpha_i$. The statement on $\beta_1$ is trivial if $\beta_1 = 0$, and is proven by passing over to $(AB)^{-1}$ if $\beta_1 > 0$.

Proof of theorem 3: First let $B$ be nonsingular, $B = C^{-1}$. Then $\lambda = a/c$ with $a \in F(A)$ and $c \in F(C)$ according to theorem 2. If $\beta_1$ and $\beta_n$ are the minimum and maximum eigenvalues of $B$, then $\beta_1^{-1}$ and $\beta_n^{-1}$ are the maximum and minimum eigenvalues of the positive definite matrix $C$, so $F(C)$ is the interval $[\beta_n^{-1}, \beta_1^{-1}]$, hence $c^{-1} \in [\beta_1, \beta_n] = F(B)$ and $\lambda \in F(A)F(B)$. The case $\det B = 0$ may be reduced to the nonsingular case by replacing $B$ by $B + tI$, $t > 0$, $t \to 0$.

REMARK 3″: In theorem 3, the additional condition on $B$ must not be dropped. For instance if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have $F(A) = F(B) = [-1, 1]$, hence $F(A)F(B) = [-1, 1]$ which does not contain the eigenvalues $\pm i$ of $AB$. However, theorem 3 holds for arbitrary matrices if stated in a modified form involving the angular field instead of the field of values.

THEOREM 4: *Let* $0 \notin$ W(B). *If* $\lambda$ *is an eigenvalue of* AB *or of* BA, *then* $\lambda \in$ W(A) W(B).

PROOF: Since $0 \notin W(B)$, $B$ is not singular, $B = C^{-1}$. By theorem 2 we have $\lambda = a/c$ with $a \in F(A) \subseteq W(A)$ and, for some normalized $x$, we have

$$c = x^*Cx = (Cx)^*B^*(Cx) = y^*B^*y,$$

$$\bar{c} = y^*By \in W(B),\ c \neq 0,\ c^{-1} = \bar{c}/|c|^2 \in W(B).$$

It is worthwhile noting that, in general, we do not have $W(AB) \subseteq W(A)\ W(B)$: for instance whenever $A$ and $B$ are hermitian matrices which do not commute, then $AB$ is not hermitian and $W(AB)$ is not real though $W(A)\ W(B)$ is real.

Theorem 4 may be used to obtain information concerning the signs of the real parts of eigenvalues.

THEOREM 5: *Let* C = AB, *where* A *is hermitian and* F(B) *is contained in the interior of the right half plane. Let* $c_+, c_0, c_-$ *denote the number of positive, vanishing, and negative real parts of the eigenvalues of* C, *and let* $a_+, a_0, a_-$ *denote the number of positive, vanishing, and negative eigenvalues of* A. *Then* $c_+ = a_+$, $c_0 = a_0$, *and* $c_- = a_-$.

The assumption on $B$ is equivalent to each of the following: $F(B)$ is contained in the right half plane; $B + B^*$ is positive definite; or $B = H + iG$, with $G$ and $H$ hermitian and $H$ positive definite.

COROLLARY 5′: *Let* A *be hermitian and* B *hermitian and positive definite. Then* AB *has as many positive, vanishing and negative eigenvalues as* A.[1]

[1] This corollary is not believed to be new, though the author has not been able to find it in the literature. It may be proved alternatively by applying the law of inertia to the matrices $A$ and $TAT^* = TABT^{-1}$, where $T^*T = B$. Conversely, the law of inertia is a special case of the corollary.

PROOF OF THEOREM 5: $B$ is nonsingular since $0 \notin F(B)$. Let $\lambda$ be an eigenvalue of $AB$. By theorem 4 we have $\lambda \in W(A)W(B)$.

(a) Let us first assume $A$ to be positive definite. Then $AB$ is nonsingular and if $\gamma \in W(A)W(B) = W(B)$, then Re $\gamma > 0$. This means $c_+ = n = a_+$, $c_0 = 0 = a_0$, $c_- = 0 = a_-$. The proof is similar in case $A$ is negative definite.

(b) Let us now treat the case of an indefinite, nonsingular $A$. Again every eigenvalue of $AB$ is nonzero, but now $W(A)W(B)$ is the union of two opposite angular domains. We change the given matrix $B$ continuously into $\beta I$, where $\beta$ is some fixed value contained in $F(B)$, by letting $B(t) = t\beta I + (1-t)B$, $0 \leq t \leq 1$. The assumptions of case (b) are satisfied for $A$ and $B(t)$. Thus we know the eigenvalues of $AB(t)$ never pass through 0. On the other hand, they vary continuously with $t$ and never leave the domain $W(B) \cup (-W(B))$. Hence $W(B)$ contains as many eigenvalues of $AB(0) = AB$ as of $AB(1) = \beta A$. This means $c_+ = a_+$. Similarly $c_- = a_-$. In addition we have $c_0 = 0 = a_0$.

(c) The last case to consider is $\det A = 0$. Without loss of generality we may assume $A$ in diagonal form. Let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \det A_1 \neq 0,$$

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Here $B_4$ has $a_0$ rows and columns. We have

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix}.$$

Therefore $AB$ has $a_0$ eigenvalues equal to 0; the remaining $n - a_0$ eigenvalues of $AB$ are those of $A_1B_1$. It is clear that $F(B_1) \subseteq F(B)$, so $A_1$ and $B_1$ fulfill the assumptions of theorem 5, case (a) or (b), hence $c_+ = a_+$, $c_- = a_-$. Also we have $c_0 = n - (c_+ + c_-) = a_0$ which completes the proof.

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(Paper 77B1 & 2–379)

JOURNAL OF RESEARCH of the National Bureau of Standards—A. Mathematical Sciences
Vol. 77B, Nos. 1 & 2, January–June 1973

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